Hodge theory

lecture 4: Sobolev L^2-spaces and Rellich lemma

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Banach spaces

DEFINITION: Let M be a topological space, and $\|f\| := \sup_M |f|$ **the sup-norm on functions**. **C^0-topology**, or **uniform topology** on the space $C^0(M)$ of bounded continuous functions is topology defined by the sup-norm.

DEFINITION: A **Banach space** is a complete normed vector space.

THEOREM: A space of bounded continuous functions on M with C^0-topology is Banach.

Proof: A uniform limit of continuous functions is continuous (Weierstrass), and a limit of a Cauchy sequence of functions in $C^0(M)$ exists pointwise because $\mathbb{R}$ is complete. ∎

Stone-Weierstrass approximation theorem

DEFINITION: Let $A \subset C^0 M$ be a subspace in the space of continuous functions. We say that A **separates the points** of M if for all distinct points $x, y \in M$, there exists $f \in A$ such that $f(x) \neq f(y)$.

THEOREM: (**Stone-Weierstrass approximation theorem**) Let M be a compact manifold and $A \subset C^0 M$ be a subring separating points, and $\overline{A}$ its closure. **Then $\overline{A} = C^0 M$.**

Proof: Handouts or the next lecture. ■

Hilbert spaces (reminder)

DEFINITION: Hilbert space is a complete, infinite-dimensional Hermitian space which is second countable (that is, has a countable dense set).

DEFINITION: Orthonormal basis in a Hilbert space H is a set of pairwise orthogonal vectors $\{x_\alpha\}$ which satisfy $|x_\alpha| = 1$, and such that H is the closure of the subspace generated by the set $\{x_\alpha\}$.

THEOREM: Any Hilbert space has a basis, and all such bases are countable.

THEOREM: All Hilbert spaces are isometric.

Proof: Each Hilbert space has a countable orthonormal basis. ■

Fourier series

CLAIM: ("Fourier series") Functions $e_k(t) = e^{2\pi\sqrt{-1}\,kt}$, $k \in \mathbb{Z}$ on $S^1 = \mathbb{R}/\mathbb{Z}$ **form an orthonormal basis in the space $L^2(S^1)$** of square-integrable functions on the circle.

Proof: Orthogonality is clear from $\int_{S^1} e^{2\pi\sqrt{-1}\,kt}dt = 0$ for all $k \neq 0$ (prove it). To show that the space of Fourier polynomials $\sum_{i=-n}^{n} a_k e_k(t)$ is dense in the space of continuous functions on circle, use the Stone-Weierstrass approximation theorem, applied to the ring $R = \langle \sin(mx), \cos(nx)\rangle$ of functions obtained from real and imaginary parts of $e^{2\pi\sqrt{-1}\,kt}$. ■

DEFINITION: Fourier monomials on a torus are functions $F_{l_1,...,l_n} := \exp(2\pi\sqrt{-1}\,\sum_{i=1}^{n} l_i t_i)$, where $l_1, ..., l_n \in \mathbb{Z}$.

CLAIM: Fourier monomials **form an orthonormal basis in the space $L^2(T^n)$** of square-integrable functions on the torus T^n.

Proof: The same. ■

L^2-norms on vector spaces

THEOREM: Let V be a vector space, and g_1, g_2 two scalar products. We say that **g_1 is bounded by g_2** if for some $C > 0$, one has $g_1 \leqslant C g_2$.

EXERCISE: Prove that **this is equivalent to the continuity of the map $(V, g_2) \longrightarrow (V, g_1)$.**

REMARK: Let g_1 be bounded by g_2. **Then the identity map extends to a continuous map on the corresponding completion spaces $L^2(V, g_2) \longrightarrow L^2(V, g_1)$.**

REMARK: The topology induced by g_1 **is equivalent to topology induced by g_2 if and only if $C^{-1} g_2 \leqslant g_1 \leqslant C g_2$.**

Sobolev's L^2-norm on $C_c^\infty(\mathbb{R}^n)$

DEFINITION: Denote by $C_c^\infty(\mathbb{R}^n)$ the space of smooth functions with compact support. For each differential monomial

$$P_\alpha = \frac{\partial^{k_1}}{\partial x_1^{k_1}} \frac{\partial^{k_2}}{\partial x_2^{k_2}} ... \frac{\partial^{k_n}}{\partial x_1^{k_n}}$$

consider the corresponding partial derivative

$$P_\alpha(f) = \frac{\partial^{k_1}}{\partial x_1^{k_1}} \frac{\partial^{k_2}}{\partial x_2^{k_2}} ... \frac{\partial^{k_n}}{\partial x_1^{k_n}} f.$$

Given $f \in C_c^\infty(\mathbb{R}^n)$, one defines **the L_p^2 Sobolev's norm $|f|_p$** as follows:

$$|f|_s^2 = \sum_{\deg P_\alpha \leqslant p} \int |P_\alpha(f)|^2 \operatorname{Vol}$$

where the sum is taken over all differential monomials P_α of degree $\leqslant p$, and $\operatorname{Vol} = dx_1 \wedge dx_2 \wedge ... dx_n$ - the standard volume form.

REMARK: Same formula defines **Sobolev's L^2-norm L_p^2 on the space of smooth functions on a torus T^n.**

Sobolev's L^2-norm on a torus

CLAIM: The Fourier monomials $F_{l_1,...,l_n} := e^{2\pi\sqrt{-1}\,\sum l_i t_i}$ **are eigenvectors for the differential monomials** $P_\alpha = \frac{\partial^{k_1}}{\partial x_1^{k_1}} \frac{\partial^{k_2}}{\partial x_2^{k_2}} ... \frac{\partial^{k_n}}{\partial x_1^{k_n}}$**. Moreover,** $P_\alpha(F_{l_1,...,l_n}) = \prod_{i=1}^n (2\pi\sqrt{-1}\,k_i)^{l_i}$**.** ■

COROLLARY: The Fourier monomials are orthogonal in the Sobolev's L^2_p-metric, and

$$|F_{l_1,...,l_n}|^2_{2,p} = \sum_{k_1+...+k_n=1}^{p} \prod_{i=1}^{n} (2\pi l_i)^{2k_i}.$$

■

Weak convergence (reminder)

DEFINITION: Let $x_i \in H$ be a sequence of points in a Hilbert space H. We say that x_i **weakly converges** to $x \in H$ if for any $z \in H$ one has $\lim_i g(x_i, z) = g(x, z)$.

REMARK: Let $y(i) = \alpha_j(i)e_j$ be a sequence of points in a a Hilbert space with orthonormal basis e_i. **Then $y(i)$ converges to $y = \sum_j \alpha_j e_j$ if and only if $\lim_i \alpha_j(i) = \alpha_i$.**

CLAIM: For any sequence $\{y(i) = \sum_j \alpha_j(i)e_j\}$ of points in a unit ball, **there exists a subsequence $\{\tilde{y}(i) = \tilde{\alpha}_j(i)e_i\}$ weakly converging to $y \in H$.**

Proof: Indeed, $|\alpha_j(i)| \leqslant 1$, hence there exist a subsequence $\tilde{y}(i) = \tilde{\alpha}_j(i)x_j$ with $\tilde{\alpha}_j(i)$ converging for each j. The limit belongs to the unit ball because otherwise $\left|\sum_{j=1}^n \tilde{\alpha}_j(i)e_j\right| > 1$, which is impossible. ■

REMARK: Note that **the function $x \longrightarrow |x|$ is not continuous in weak topology.** Indeed, weak limit of $\{e_i\}$ is 0. The proof above shows that $|\cdot|$ is semicontinuous.

Compact operators (reminder)

DEFINITION: Precompact set is a set which has compact closure. **A compact operator** is an operator which maps bounded sets to precompact.

THEOREM: Let $A: H \longrightarrow H_1$ be an operator on Hilbert spaces. Then A **is compact if and only if it maps weakly convergent sequences to convergent ones.**

Rellich lemma for a torus

THEOREM: (Rellich lemma for a torus)
The identity map $L^2_p(T^n) \longrightarrow L^2_{p-1}(T^n)$. is compact.

Proof. Step 1: Consider, instead of L^2_p-metric, the metric q_p which is orthogonal in the same basis and satisfies $|F_{l_1,...,l_n}|_{q_p} := 1 + (2\pi)^p \sum_{i=1}^n l_i^p$. Clearly, $|F_{l_1,...,l_n}|_{q_p} \leqslant |F_{l_1,...,l_n}|_{2,p}$ and $|F_{l_1,...,l_n}|_{q_p} \geqslant C^{-1}|F_{l_1,...,l_n}|_{2,p}$, where C is a number of differential monomials of degree p. Therefore, **q_p and L^2_p induce the same topology,** and it would suffice to prove the Rellich lemma for the identity map $L^2(T^n, q_p) \longrightarrow L^2(T^n, q_{p-1})$.

Step 2: Now,

$$\frac{|F_{l_1,...,l_n}|^2_{q_p}}{|F_{l_1,...,l_n}|^2_{q_{p-1}}} = \frac{\sum_{i=1}^n (2\pi)^p l_i^{2p}}{\sum_{i=1}^n (2\pi)^{p-1} l_i^{2p-2}} \geqslant \frac{n}{\max l_i^2}.$$

Step 3: Let $x_i \in L^2(T^n, q_p)$ be weakly converging to x, with $|x_i|_{q_p} < 1$. Let $x_i = y_i + z_i$, with y_i being the sum of all Fourier terms with $\max |l_i| < N$, and z_i the rest. Then $|z_i - z|_{q_{p-1}} < \frac{\sqrt{n}}{N}|z_i - z|_{q_p} < \frac{2\sqrt{n}}{N}$, and y_i converges to y because it is a sum of finitely many terms which all converge. **We obtain that $\lim_i |x_i - x|_{q_{p-1}} = 0$, hence a x_i (strongly) converges to x.** ■

Franz Rellich (1906-1955)

After Weyl's resignation [from Göttingen], his former assistant, Franz Rellich, became Institute Director ... Rellich had only a low-level appointment and ... was not an established figure ... There was need for a prominent mathematical figure who was suitable politically to take over the leadership in Gottingen. Furthermore, in mid-December, Rellich was ordered to report on January 7 for ten weeks to a field-sports camp near Berlin. This was, in fact, a mistake, since Rellich, as an Austrian citizen, was not subject to such forced training regimens. When he arrived at the camp, he was not admitted on these grounds. However, on December 27, the Curator had, after some hesitation, replaced Rellich with Werner Weber as acting director of the Mathematical Institute. Rellich himself would lose his position at Gottingen six months later, on June 18. – S. L. Segal, *Mathematicians under the Nazis*

Rellich lemma for $C^\infty_K(\mathbb{R}^n)$

COROLLARY: Let $C^\infty_K(\mathbb{R}^n)$ be the space of smooth functions on $\mathbb{R}^n$ with support in a compact set K. **Then the identity map**

$$L^2_p(C^\infty_K(\mathbb{R}^n)) \longrightarrow L^2_{p-1}(C^\infty_K(\mathbb{R}^n))$$

is compact.

Proof: We consider a quotient map $\mathbb{R}^n \longrightarrow T^n$ which is bijective on K for an appropriate choice of a lattice. This embeds $C^\infty_K(\mathbb{R}^n)$ to $C^\infty(T^n)$, and this embedding is compatible with the L^2_p-norms. ■

Sobolev's L^2-norm on a compact manifold

DEFINITION: Let M be a manifold, $\{U_i\}$ a finite atlas, and $\{\psi_i\}$ the corresponding partition of unity. We will identify U_i with bounded subsets in $\mathbb{R}^n$. Given a function $f \in C^\infty(M)$, define **the Sobolev L^2_p-metric** $|f|^2_{2,p}$ as $\sum |f\psi_i|^2_{2,p}$, where $f\psi_i$ is considered as a function with compact support on $U_i \subset \mathbb{R}^n$, and $\cdot|_{2,p}$ is the Sobolev L^2_p-metric on $C^\infty_c(\mathbb{R}^n)$.

PROPOSITION: The topology induced on $C^\infty(M)$ by L^2_p **is independent from the choice of $\{U_i\}$ and $\{\psi_i\}$.**

Proof: Let $\Psi: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a map with uniformly bounded partial derivatives up to p-th. From the definition of the L^2_p-norm and the chain rule it follows that

$$C^{-1}|f|^q_{2,p} \leqslant |\Psi^* f|^q_{2,p} \leqslant C|f|^q_{2,p}$$

where the constant C depends on the supremum of partial derivatives of Ψ. Then, for any refinement $\{V_j\}$ of $\{U_i\}$ and the corresponding partition of unity $\{\varphi_j\}$, the L^2_p-norm of $f\psi_i$ associated with $\{V_j, \varphi_j\}$ is bounded by the one associated with $\{U_i, \psi_i\}$. For the same reason the L^2_p-norm of $f\varphi_j$ associated with $\{V_j, \varphi_j\}$ is bounded by the one associated with $\{U_i, \psi_i\}$. This gives an estimate of form $C^{-1}g_2 \leqslant g_1 \leqslant Cg_2$ for L^2_p-metrics associated with a cover and its refinement. To obtain a similar estimate for two different covers, we find a common refinement. ■

Rellich lemma for $C^\infty(M)$.

THEOREM: (Rellich lemma) Let M be a compact manifold. **Then the identity map $L^2_p(M) \longrightarrow L^2_{p-1}(M)$ is compact.**

Proof. Step 1: Let $\{U_i\}$ be a finite atlas on M and $\{\psi_i\}$ the corresponding partition of unity. We will identify U_i with bounded subsets in $\mathbb{R}^n$. Then $|f|^2_{2,p} = \sum_i |\psi_i f|^2_{2,p}$, where the second $|\cdot|^2_{2,p}$-norm is taken on a bounded subset in $\mathbb{R}^n$.

Step 2: Let $f_j \in L^2_p(M)$ be a sequence weakly converging to f. Then $\psi_i f_j$ weakly converges to a function $\tilde{f}_i$ with support in Supp(ψ_i). Using Rellich lemma for functions on $\mathbb{R}^n$ with compact support, we obtain that $\psi_i f_j$ converges in L^2_{p-1} to $\tilde{f}_i$. Then $f_j = \sum_i \psi_i f_j$ converges in L^2_{p-1} to $\sum_i \tilde{f}_i$. ■