Hodge theory

lecture 5: Fredholm operators

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Operators with closed image

THEOREM: ("Banach inverse mapping theorem")
Let $A: H_1 \longrightarrow H_2$ be an operator on Hilbert spaces with closed image. **Then** $A^{-1}: \operatorname{im} A \longrightarrow H_1/\ker A$ **is continuous.**

Proof. Step 1: Suppose that A^{-1} is not continuous. This means that $\sup_{x\in \operatorname{im} A} \frac{|A^{-1}(x)|}{|x|} = \infty$, equivalently, that $\inf_{x\in(\ker A)^\perp} \frac{|A(x)|}{|x|} = 0$. In other words, **there exists a sequence of vectors** $v_1, ..., v_n, ... \in (\ker A)^\perp$ **such that** $|A(v_i)| = 0$.

Step 2: Choose a subsequence $w_1, ..., w_n, ...$ in $\{v_i\}$ such that $|w_i| = 1$ and $\sum |A(w_i)|^2 < \infty$. Restrict A to the space W generated by w_i. Clearly, the vector $\sum A(w_i)$ has no preimage, hence A is never surjective.

Step 3: Replacing H_1 by $H_1/\ker A$ and H_2 by $\operatorname{im} A$, we may assume that $\ker A = 0$, $\operatorname{im} A = H_2$. Clearly, $x \perp A(W) \Leftrightarrow A^*(x) \perp W \Leftrightarrow A^*(x) \in W^\perp$ and $x \perp A(W^\perp) \Leftrightarrow A^*(x) \in W$. Applying $(A^*)^{-1}$ to the direct sum decomposition $H_1 = W \oplus W^\perp$, **we obtain the direct sum decomposition** $H_2 = \overline{A(W)} \oplus \overline{A(W^\perp)}$. Since A cannot be surjective onto $\overline{A(W)}$ (Step 2), this leads to the contradiction. ■

Fredholm operators

DEFINITION: A continuous operator $F: H_1 \longrightarrow H_2$ of Hilbert spaces is called **Fredholm** if its image is closed and kernel and cokernel are finite-dimensional.

REMARK: "Cokernel" of a morphism $F: H_1 \longrightarrow H_2$ of topological vector spaces is often defined as $\frac{H_2}{\overline{\operatorname{im} F}}$.

DEFINITION: An operator $F: H_1 \longrightarrow H_2$ **has finite rank** if its image has finite rank.

CLAIM: An operator $F: H_1 \longrightarrow H_2$ **is Fredholm** if and only if there exists $F_1: H_2 \longrightarrow H_1$ such that **the operators $\operatorname{Id} - FF_1$ and $\operatorname{Id} - F_1 F$ have finite rank.**

Proof: This is because F defines an isomorphism $F: H_1 / \ker F \longrightarrow \operatorname{im} F$ as shown above. ■

Erik Ivar Fredholm
Born: 7 April 1866 in Stockholm, Sweden
Died: 17 August 1927 in Danderyd, County of Stockholm, Sweden

...As a young boy he played the flute, but later took up playing the violin. He particularly loved to play Bach. Again he combined his talents, applying his mechanical skills to music as well as to mathematics. Unlikely as it sounds, he built his first violin from half a coconut, while he also used his talents at building machines to make one to solve differential equations...

Fredholm operators: invertible up to a finite-dimensional summand

REMARK: For any operator $F: H_1 \longrightarrow H_2$ on Hilbert spaces, one has $\ker F^* = H_2/\overline{\operatorname{im} F}$. Also, F is Fredholm if and only if F^* is Fredholm, because **Fredholm is the same as "invertible after adding a finite-dimensional subspace and extending the operator to this finite-dimensional subspace in appropriate way".**

Lemma 1: Let F, G be operators on a Hilbert space H, with $F \circ G$ Fredholm and $G \circ F$ Fredholm. **Then F and G are Fredholm.**

Proof: Consider a category $\mathcal{C}$ with $\mathcal{O}\lfloor(\mathcal{C})$ Hilbert spaces and $\mathcal{M}\wr\nabla(H_1, H_2) = \operatorname{Hom}(H_1, H_2)/\mathcal{F}$ where $\mathcal{F}$ is the space of finite rank maps. Clearly, a morphism is invertible in $\mathcal{C}$ if and only if it is Fredholm. Then FG and GF are invertible in $\mathcal{C}$, giving $FGP = \operatorname{Id}$ and $PGF = \operatorname{Id}$ for some morphism P. This gives $F \circ GP = \operatorname{Id}$ and $PG \circ F = \operatorname{Id}$, hence F is invertible in $\mathcal{C}$. ■

Operator norm

DEFINITION: Let V_1, V_2 be vector spaces equipped with a norm. **Norm** ("operator norm") of a linear operator $E: V_1 \longrightarrow V_2$ is the number $\|E\| := \sup_{|v \neq 0} \frac{|E(v)|}{|v|}$.

REMARK: An operator is continuous if and only if its norm is finite.

CLAIM: The space $\operatorname{Hom}(V, W)$ of operators on Banach spaces with the operator norm **is a Banach space. It is not a Hilbert space** even when V, W are Hilbert spaces.

Limits of operators of finite rank

REMARK: Hausdorff limit of compacts is again compact. Therefore, **compact operators are closed in norm topology**.

THEOREM: Let U, W be Hilbert spaces, and $K \subset \operatorname{Hom}(U, W)$ be the closure of the space of operators of finite rank in operator norm topology. Then K **is the space of compact operators.**

Proof. Step 1: Since finite rank operators are compact, and compact operators are closed in operator norm topology, K **is contained in the space of compact operators.**

Step 2: Fix a compact operator $A: U \longrightarrow W$. Let $U_0 \subset U_1 \subset ...$ be a sequence of finite-dimensional spaces such that $U = \overline{\bigcup U_i}$, and $A_i \in \operatorname{Hom}(U, W)$ be operator equal to A on U_i and to 0 on $U_i^\perp$. Then $\|A - A_i\| = \sup_{x \in U_i^\perp} \frac{|A_i(x)|}{|x|}$. **Unless** $\lim_i \|A - A_i\| = 0$**, we have an infinite sequence** x_i **of orthogonormal vectors in** W **such that** $|A(x_i)| \geqslant \varepsilon > 0$**,** which is impossible because A is compact. Then $A = \lim A_i$. ■

Fredholm operators and compact operators

THEOREM: The set of Fredholm operators is open in the operator norm topology.

Proof. Step 1: Let $F: U \longrightarrow V$ be a Fredholm operator, and $U_1 := (\ker F)^\perp$. Since F is invertible on U_1, it satisfies $\inf_{x \in U_1} \frac{|F(x)|}{|x|} > 2\varepsilon$. Then, for any operator A with $\|A\| < \varepsilon$, one has $\inf_{x \in U_1} \frac{|F+A(x)|}{|x|} > \varepsilon$. **This implies that $F\big|_{U_1}$ is an invertible map to its image, which is closed.** In particular, $\ker(F+A)$ is finite-dimensional.

Step 2: To obtain that $\operatorname{coker}(F+A)$ is finite-dimensional for $\|A\|$ sufficiently small, we observe that $\operatorname{coker}(F+A) = \ker(F^* + A^*)$, and F^* is also Fredholm. Then Step 1 implies that $\ker(F^*+A^*)$ **is finite-dimensional for $\|A\|$ sufficiently small.** ■

COROLLARY: Let A be compact and F Fredholm. **Then $A+F$ is Fredholm.**

Proof: Let A_i be a sequence of operators with finite rank converging to A. Then $F + (A - A_i)$ is Fredholm for i sufficiently big, because the set of Fredholm operators is open. However, a sum of Fredholm operator and operator of finite rank is Fredholm, hence $F + A = F + (A - A_i) + A_i$ is also Fredholm. ■

Calkin algebra

CLAIM: Compact operators form a two-sided ideal in the algebra $\operatorname{Hom}(H, H)$.

Proof: Under a continuous homomorphism of normed vector spaces, **an image of a bounded set is bounded, and an image of a compact set is compact.** Therefore, for any compact $K \in \operatorname{Hom}(H, H)$ and any $A \in \operatorname{Hom}(H, H)$, the operators KA and AK are compact. ■

DEFINITION: Calkin algebra is the quotient of $\operatorname{Hom}(H, H)$ by the ideal of compact operators.

THEOREM: An operator $F: H_1 \longrightarrow H_2$ **is Fredholm if and only if its image in the Calkin algebra is invertible.**

Proof. Step 1: Invertibility in Calkin algebra means that there exists $G: H_2 \longrightarrow H_1$ such that **the operators $\operatorname{Id} - FG$ and $\operatorname{Id} - GF$ are compact.**

Step 2: By definition, F is Fredholm if and only if there exists $G: H_2 \longrightarrow H_1$ such that $\operatorname{Id} - FG$ and $\operatorname{Id} - GF$ have finite rank. **Therefore, F is invertible in the Calkin algebra.**

Step 3: Assume that $\operatorname{Id} - FG$ and $\operatorname{Id} - GF$ are compact. A sum of Fredholm and compact is compact, hence FG and GF are Fredholm. Then F and G are Fredholm by Lemma 1. ■

Index of Fredholm operators

DEFINITION: Let $F: H \longrightarrow H_1$ be a Fredholm operator. Define its **index** as $\operatorname{ind}(F) := \dim\ker F - \dim\operatorname{coker} F$.

CLAIM: Let $H_1 \xrightarrow{F} H_2 \xrightarrow{G} H_3$ be Fredholm operators. **Then** $\operatorname{ind}(FG) = \operatorname{ind}(F) + \operatorname{ind}(G)$**.**

Proof: For F, G finite-dimensional this is clear, because then $\operatorname{ind}(F) = \dim H_1 - \dim H_2$, $\operatorname{ind}(G) = \dim H_2 - \dim H_3$ and $\operatorname{ind}(FG) = \dim H_1 - \dim H_3$. To reduce everything to the finite-dimensional case, chose $W_1 \subset H_1, W_2 \subset H_2, W_3 \subset H_3$ of finite codimension. Then $\operatorname{ind}(FG)$ is index of the map $H_1/W_1 \xrightarrow{FG} H_3/W_3$, and $\operatorname{ind} F$, $\operatorname{ind} G$ are indices of the maps $H_1/W_1 \xrightarrow{F} H_2/W_2 \xrightarrow{G} H_3/W_3$. ■

CLAIM: Let $F: H \longrightarrow H_1$ be a Fredholm operator, and $A: H \longrightarrow H_1$ a finite rank operator. **Then** $\operatorname{ind}(F+A) = \operatorname{ind}(F)$**.**

Proof: Find a closed subspace $W \subset \ker A$ of finite codimension such that $F|_W \longrightarrow F(W)$ is an isomorphism. Then $\operatorname{ind}(F+A)$ is index of the map $F+A: H/W \longrightarrow H_1/F(W)$, and $\operatorname{ind}(F)$ is index of $F: H/W \longrightarrow H_1/F(W)$. However, for finite-dimensional spaces P, Q and a map $G: P \longrightarrow Q$, $\operatorname{ind}(G) = \dim P - \dim Q$, hence $\operatorname{ind}(F+A) = \operatorname{ind}(F)$. ■

Index of a sum of Fredholm operator and compact

COROLLARY: For any compact operator $K: H \longrightarrow H$, one has $\operatorname{ind}(K + \operatorname{Id}) = 0$.

Proof: Decompose K onto a sum $K = K_1 + K_0$, with K_1 of finite rank, and $\|K_0\| < 1$. Then $(\operatorname{Id} + K_0)^{-1} = \operatorname{Id} - K_0 + K_0^2 - ...$, hence $\operatorname{ind}(\operatorname{Id} + K_0) = 0$, and $\operatorname{ind}(\operatorname{Id} + K) = 0$ because $\operatorname{Id} + K$ is a sum of $\operatorname{id} + K_0$ and an operator of finite rank. ■

THEOREM: Let F be a Fredholm operator, and K compact. **Then $\operatorname{ind}(F + K) = \operatorname{ind}(F)$.**

Proof: There exists a Fredholm operator G such that $FG = \operatorname{Id} + A$, $GF = \operatorname{Id} + B$, and A, B are compact operators. Then $(F + K)G = \operatorname{Id} + A - KG$, which is a sum of identity and a compact operator, giving $\operatorname{ind}((F + K)G) = 0$. Using $\operatorname{ind}(FG) = \operatorname{ind}(F) + \operatorname{ind}(G)$, we obtain $\operatorname{ind}(F) = -\operatorname{ind}(G)$ and

$$0 = \operatorname{ind}((F + K)G) = \operatorname{ind}(G) + \operatorname{ind}(F + K)$$

giving $\operatorname{ind}(F + K) = -\operatorname{ind}(G) = \operatorname{ind}(F)$. ■

Index of Fredholm operator is locally constant

REMARK: Recall that the set of Fredholm operators **is open in the operator norm topology.**

COROLLARY: Index is a locally constant function on the space of Fredholm operators taken with the operator norm topology.

Proof: Let F, G be Fredholm, $FG = \operatorname{Id} + A$, $GF = \operatorname{Id} + B$, where A, B are compact, and $\|H\| < \varepsilon$. Then $(F+H)G = \operatorname{Id} + A + HG$. Choosing ε in such a way that $\|HG\| < 1$, we obtain that $\operatorname{Id} + HG$ is invertible, hence the index of $\operatorname{Id} + A + HG$ is zero (it is a sum of invertible and a compact operator). This gives

$$0 = \operatorname{ind}((F+H)G) = \operatorname{ind}(F+H) + \operatorname{ind}(G) = \operatorname{ind}(F+H) - \operatorname{ind}(F)$$

hence $\operatorname{ind}(F+H) = \operatorname{ind}(F)$. ■