Hodge theory

lecture 6: Laplace operator is Fredholm

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Fredholm operators (reminder)

DEFINITION: A continuous operator $F: H_1 \longrightarrow H_2$ of Hilbert spaces is called **Fredholm** if its image is closed and kernel and cokernel are finite-dimensional.

REMARK: "Cokernel" of a morphism $F: H_1 \longrightarrow H_2$ of topological vector spaces is often defined as $\frac{H_2}{\overline{\operatorname{im} F}}$.

DEFINITION: An operator $F: H_1 \longrightarrow H_2$ **has finite rank** if its image has finite rank.

CLAIM: An operator $F: H_1 \longrightarrow H_2$ **is Fredholm** if and only if there exists $F_1: H_2 \longrightarrow H_1$ such that **the operators $\operatorname{Id} - FF_1$ and $\operatorname{Id} - F_1F$ have finite rank.**

Proof: This is because F defines an isomorphism $F: H_1/\ker F \longrightarrow \operatorname{im} F$ as shown above. ∎

Fredholm operators and compact operators (reminder)

THEOREM: The set of Fredholm operators is open in the operator norm topology.

Proof. Step 1: Let $F: U \longrightarrow V$ be a Fredholm operator, and $U_1 := (\ker F)^\perp$. Since F is invertible on U_1, it satisfies $\inf_{x\in U_1} \frac{|F(x)|}{|x|} > 2\varepsilon$. Then, for any operator A with $\|A\| < \varepsilon$, one has $\inf_{x\in U_1} \frac{|F+A(x)|}{|x|} > \varepsilon$. **This implies that $F\big|_{U_1}$ is an invertible map to its image, which is closed.** In particular, $\ker(F+A)$ is finite-dimensional.

Step 2: To obtain that $\operatorname{coker}(F+A)$ is finite-dimensional for $\|A\|$ sufficiently small, we observe that $\operatorname{coker}(F+A) = \ker(F^*+A^*)$, and F^* is also Fredholm. Then Step 1 implies that $\ker(F^*+A^*)$ **is finite-dimensional for $\|A\|$ sufficiently small.** ∎

COROLLARY: Let A be compact and F Fredholm. **Then $A+F$ is Fredholm.**

Proof: Let A_i be a sequence of operators with finite rank converging to A. Then $F+(A-A_i)$ is Fredholm for i sufficiently big, because the set of Fredholm operators is open. However, a sum of Fredholm operator and operator of finite rank is Fredholm, hence $F+A = F+(A-A_i)+A_i$ is also Fredholm. ∎

Equivalent scalar products on vector spaces

THEOREM: Let V be a vector space, and g_1, g_2 two scalar products. We say that g_1 **is equivalent to** g_2 if these two scalar product induce the same topology.

THEOREM: The topology induced by g_1 **is equivalent to topology induced by** g_2 **if and only if** $C^{-1}g_2 \leqslant g_1 \leqslant Cg_2$ for some $C > 0$.

Proof: Consider the identity operator $A: (V, g_1) \longrightarrow (V, g_2)$. Its operator norm is $\sup_{x\neq 0} \frac{g_2(x,x)}{g_1(x,x)}$. Operator norm is bounded if and only if Id is continuous, and this is equivalent to existence of a constant $C > 0$ such that $C^{-1}g_2 \leqslant g_1$. Existence of a constant C such that $g_1 \leqslant Cg_2$ is equivalent to continuity of A^{-1}. ■

Equivalent scalar products and symmetric operators

LEMMA: Let V be a vector space, and g, g_1 scalar products. Consider the symmetric operator B_1 such that $g_1(x,y) = g(B_1(x), y)$. **Then**

$$\sup_x \frac{g(B_1(x), B_1(x))}{g(x,x)} = \left(\sup_x \frac{g_1(x,x)}{g(x,x)}\right)^2.$$

Proof: By Cauchy-Schwarz, $g(x,x)g(B_1(x), B_1(x)) \geqslant g(B_1(x), x)^2 = g_1(x,x)^2$. This gives $\sup_x \frac{g(B_1(x),B_1(x))}{g(x,x)^2} \geqslant \left(\sup_x \frac{g_1(x,x)}{g(x,x)}\right)^2$. On the other hand, $\sup_x \frac{g(B_1(x),B_1(x))}{g(x,x)}$ is norm of B_1^2, which gives

$$\sup_x \frac{g(B_1(x), B_1(x))}{g(x,x)} = \|B_1^2\| \leqslant \|B_1\|^2 = \sup_x \left(\frac{g_1(x,x)}{g(x,x)}\right)^2$$

hence $\sup \frac{g(B_1(x),B_1(x))}{g(x,x)^2} \leqslant \left(\sup_x \frac{g_1(x,x)}{g(x,x)}\right)^2$. ■

Equivalent scalar products and Fredholm operators

REMARK: A continuous operator $F: H_1 \longrightarrow H_2$ in vector spaces with scalar product is called **Fredholm** if it is Fredholm on their completions (which are Hilbert spaces).

Corollary 1: Let g, g_1, g_2 be metrics on V, and consider the symmetric operators B_i such that $g_i(x,y) = g(B_i(x), y)$. Denote by $\tilde{g}_2$ the metric $\tilde{g}_2(x,y) := g_2(B_2(x), B_2(y))$. **Then g_1 is equivalent to g_2 if and only if $B_1: (V, \tilde{g}_2) \longrightarrow (V, g)$ is Fredholm.**

Proof: $B_1: (V, \tilde{g}_2) \longrightarrow (V, g)$ is Fredholm if and only if it for some constant $C > 0$, one has $C^{-1} g(B_2(x), B_2(x)) \leqslant g(B_1(x), B_1(x)) \leqslant Cg(B_2(x), B_2(x))$. This is the same as $C^{-1} g_2(x,x) \leqslant g_1(x,x) \leqslant Cg_2(x,x)$ by the previous lemma.

■

Sobolev's L^2-norm on $C_c^\infty(\mathbb{R}^n)$ (reminder)

DEFINITION: Denote by $C_c^\infty(\mathbb{R}^n)$ the space of smooth functions with compact support. For each differential monomial

$$P_\alpha = \frac{\partial^{k_1}}{\partial x_1^{k_1}} \frac{\partial^{k_2}}{\partial x_2^{k_2}} \cdots \frac{\partial^{k_n}}{\partial x_1^{k_n}}$$

consider the corresponding partial derivative

$$P_\alpha(f) = \frac{\partial^{k_1}}{\partial x_1^{k_1}} \frac{\partial^{k_2}}{\partial x_2^{k_2}} \cdots \frac{\partial^{k_n}}{\partial x_1^{k_n}} f.$$

Given $f \in C_c^\infty(\mathbb{R}^n)$, one defines **the L_p^2 Sobolev's norm $|f|_p$** as follows:

$$|f|_s^2 = \sum_{\deg P_\alpha \leqslant p} \int |P_\alpha(f)|^2 \operatorname{Vol}$$

where the sum is taken over all differential monomials P_α of degree $\leqslant p$, and $\operatorname{Vol} = dx_1 \wedge dx_2 \wedge \ldots dx_n$ - the standard volume form.

REMARK: Same formula defines **Sobolev's L^2-norm L_p^2 on the space of smooth functions on a torus T^n.**

Connections (reminder)

DEFINITION: Recall that **a connection** on a bundle B is an operator $\nabla : B \longrightarrow B \otimes \Lambda^1 M$ satisfying $\nabla(fb) = b \otimes df + f\nabla(b)$, where $f \longrightarrow df$ is de Rham differential. When X is a vector field, we denote by $\nabla_X(b) \in B$ the term $\langle \nabla(b), X \rangle$.

REMARK: In local coordinates, connection on B is a sum of differential and a form $A \in \operatorname{End} B \otimes \Lambda^1 M$. Therefore, ∇_X is a derivation along X plus linear endomorphism. This implies that **any first order differential operator on B is expressed as a linear combination of the compositions of covariant derivatives ∇_X and linear maps.**

This follows from the definition of the first order differential operator: **by definition, it is a linear combination of partial derivatives combined with linear maps.**

Connection and a tensor product (reminder)

REMARK: A connection ∇ on B gives a connection $B^* \xrightarrow{\nabla^*} \Lambda^1 M \otimes B^*$ on the dual bundle, by the formula

$$d(\langle b, \beta\rangle) = \langle \nabla b, \beta\rangle + \langle b, \nabla^* \beta\rangle$$

These connections are usually denoted **by the same letter ∇.**

REMARK: For any tensor bundle $\mathcal{B}_1 := B^* \otimes B^* \otimes ... \otimes B^* \otimes B \otimes B \otimes ... \otimes B$ **a connection on B defines a connection on $\mathcal{B}_1$** using the Leibniz formula:

$$\nabla(b_1 \otimes b_2) = \nabla(b_1) \otimes b_2 + b_1 \otimes \nabla(b_2).$$

L^2_p-metrics and connections

DEFINITION: Let F be a vector bundle on a compact manifold. The **L^2_p-topology** on the space of sections of F is a topology defined by the norm $|f|_p$ with $|f|_p^2 = \sum_{i=0}^p \int_M |\nabla^i f|^2 \operatorname{Vol}_M$, for some connection and scalar product on F and $\Lambda^1 M$.

REMARK: **The metric $|f|_p^2$ is equivalent to the Sobolev's L^2_p-metric on $C^\infty(M)$.** Indeed, all partial derivatives of a function f are expressed through $\nabla^i f$, hence an L^2-bound on partial derivatives gives L^2-bound on $\nabla^i f$, and is given by such a bound.

From now on, **we write (x,y) instead of $\int_M (x,y) \operatorname{Vol}_M$.** This metric is also denoted L^2; the space of sections of B with this metric (B, L^2).

DEFINITION: We define the **Sobolev's L^2_p-metric on vector bundles** by $L^2_p(x,y) = \sum_{i=0}^p (\nabla^i(x), \nabla^i(y))$.

L^2_p-metrics and Fredholm maps

First, let's show that we can drop all terms in this sum, except two.

Theorem 1: The Sobolev's L^2_p-metric is equivalent to $g(x,y) := (\nabla^p(x), \nabla^p(y)) + (x,y)$.

Proof. Step 1: Let $D_1 = \sum_{i=0}^p \nabla^i$ mapping B to $\left(\bigoplus_{i=0}^p (\Lambda^1)^{\otimes p}\right) \otimes B$ and $D_2(x) = \nabla^p + x$ mapping B to $(\Lambda^1 M)^{\otimes p} \otimes B \oplus B$. Then $L^2_p(x,y) = (D_1(x), y)$ and $g(x,y) = (D_2(x), y)$. Notice that $L^2_p(x,y) = (D_1^* D_1 x, y)$ and $g(x,y) = (D_2^* D_2 x, y)$.

Step 2: To prove that these two metrics are equivalent, we need to show that $D_2^* D_2 : (B,h) \longrightarrow (B, L^2)$ is Fredholm, where $h(x,y) = (D_1^* D_1 x, D_1^* D_1 y)$ (Corollary 1).

Step 3: On a flat torus, the metric h is equivalent to L^2_{2p}. Using the same argument as proves the Rellich lemma, we obtain that any differential operator Φ of order $< 2p$ defines a compact operator $\Phi : (B,h) \longrightarrow (B, L^2)$.

Step 4: The map $D_1^* D_1 : (B,h) \longrightarrow (B, L^2)$ is by definition an isometry, and $D_1^* D_1 - D_2^* D_2$ is a differential operator of lower order, which is compact as a map $(B,h) \longrightarrow (B, L^2)$ by the Rellich lemma. Then $D_2^* D_2 - D_1^* D_1$ is a compact operator, and $D_2^* D_2$ is Fredholm whenever $D_1^* D_1$ is Fredholm. ■

L^2_p-metrics and symbols of elliptic operators

The same argument proves the following result.

THEOREM: Let B be a vector bundle, and $D: B \longrightarrow B$ a differential operator which has the same symbol as $(\nabla^p)^*\nabla^p$. **Then $D: (B, L^2_{2p}) \longrightarrow (B, L^2)$ is Fredholm.**

Proof. Step 1: Denote by U the differential operator $(\nabla^{2p})^*\nabla^{2p}$. To show that $D: (B, L^2_{2p}) \longrightarrow (B, L^2)$ is Fredholm, it would suffice to prove that the metric $(x,y) + (D(x), D(y))$ is equivalent to $L^2_{2p}(x,y)$. The L^2_{2p}-metric is equivalent to $(x,y) + (U(x), y)$, as shown in Theorem 1.

Step 2: For any two differential operators A, B, symbol of AB is equal to the symbol of BA. Therefore, the symbol of $U = (\nabla^{2p})^*\nabla^{2p}$ is equal to the symbol of $(\nabla^p)^*\nabla^p(\nabla^p)^*\nabla^p$ and this is equal to the symbol of D^*D. This implies that $U - D^*D$ is an operator of order less than $2p$, hence defines a compact map $(B, L^2_{2p}) \longrightarrow (B, L^2)$. Therefore, the metric $(x,y) + (D^*Dx, y)$ is equivalent to $(x,y) + (Ux, y)$ which is equivalent to L^2_{2p}-metric, as shown in Theorem 1. ■

Laplace operators

DEFINITION: Let M be a Riemannian manifold, and $d: \Lambda^*(M) \longrightarrow \Lambda^{*+1}(M)$ de Rham differential. Then $dd^* + d^*d$ is called **the Laplacian**.

DEFINITION: Let M be a Riemannian manifold, and B a bundle with orthogonal metric and a connection $\nabla: B \longrightarrow B \otimes \Lambda^1 M$. Using the formula $\nabla(b \otimes \eta) = \nabla(b) \wedge \eta + b \otimes d\eta$, we extend ∇ to an operator $\nabla: B \otimes \Lambda^i M \longrightarrow B \otimes \Lambda^{i+1} M$ satisfying the Leibnitz equation. This operator is denoted d_∇ to distinguish it from the connection. **The Laplacian with coefficients in B** is $d_\nabla d_\nabla^* + d_\nabla^* d_\nabla$.

THEOREM: The Laplacian has the same symbol $\sigma \in$ Sym$^2(TM) \otimes$ End$(\Lambda^* M \otimes B)$ as $\nabla^*\nabla$, and it is equal to $g^{-1} \otimes \mathrm{Id}_{B \otimes \Lambda^* M}$, where $g^{-1} \in$ Sym$^2 TM$ is the bivector which corresponds to the Riemannian metric.

We shall prove it next week. The following corollary is immediate.

COROLLARY: The Laplacian is a Fredholm map from $(\Lambda^*(M) \otimes B, L^2_p)$ to $(\Lambda^*(M) \otimes B, L^2_{p-2})$.

Proof: Indeed, **Laplacian is a sum of a Fredholm map $(\nabla^*)\nabla$ and a compact operator** (all lower order differential operators are compact by Rellich lemma). ∎