Hodge theory

lecture 7: Weitzenböck formula

NRU HSE, Moscow

Misha Verbitsky, February 14, 2018

REMINDER: de Rham algebra

DEFINITION: Let $\Lambda^* M$ denote the vector bundle with the fiber $\Lambda^* T_x^* M$ at $x \in M$ ($\Lambda^* T^* M$ is the Grassmann algebra of the cotangent space $T_x^* M$). The sections of $\Lambda^i M$ are called **differential i-forms**. The algebraic operation "wedge product" defined on differential forms is $C^\infty M$-linear; the space $\Lambda^* M$ of all differential forms is called **the de Rham algebra**.

REMARK: $\Lambda^0 M = C^\infty M$.

THEOREM: **There exists a unique operator $C^\infty M \xrightarrow{d} \Lambda^1 M \xrightarrow{d} \Lambda^2 M \xrightarrow{d} \Lambda^3 M \xrightarrow{d} ...$ satisfying the following properties**

1. On functions, d is equal to the differential.
2. $d^2 = 0$
3. $d(\eta \wedge \xi) = d(\eta) \wedge \xi + (-1)^{\tilde\eta} \eta \wedge d(\xi)$, where $\tilde\eta = 0$ where $\eta \in \lambda^{2i} M$ is **an even form,** and $\eta \in \lambda^{2i+1} M$ is **odd.**

DEFINITION: The operator d is called **de Rham differential**.

DEFINITION: A form η is called **closed** if $d\eta = 0$, **exact** if $\eta in \operatorname{im} d$. The group $\frac{\ker d}{\operatorname{im} d}$ is called **de Rham cohomology** of M.

Supercommutator (reminder)

DEFINITION: A **supercommutator** of pure operators on a graded vector space is defined by a formula $\{a,b\} = ab - (-1)^{\tilde a \tilde b} ba$.

DEFINITION: A graded associative algebra is called **graded commutative** (or "supercommutative") if its supercommutator vanishes.

EXAMPLE: The Grassmann algebra is supercommutative.

DEFINITION: A graded Lie algebra (Lie superalgebra) is a graded vector space $\mathfrak{g}^*$ equipped with a bilinear graded map $\{\cdot,\cdot\}: \ \mathfrak{g}^* \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ which is graded anticommutative: $\{a,b\} = -(-1)^{\tilde a \tilde b}\{b,a\}$ and satisfies **the super Jacobi identity** $\{c,\{a,b\}\} = \{\{c,a\},b\} + (-1)^{\tilde a \tilde c}\{a,\{c,b\}\}$

EXAMPLE: Consider the algebra $\operatorname{End}(A^*)$ of operators on a graded vector space, with supercommutator as above. **Then $\operatorname{End}(A^*), \{\cdot,\cdot\}$ is a graded Lie algebra.**

Lemma 1: Let d be an odd element of a Lie superalgebra, satisfying $\{d,d\} = 0$, and L an even or odd element. **Then $\{\{L,d\},d\} = 0$.**

Proof: $0 = \{L,\{d,d\}\} = \{\{L,d\},d\} + (-1)^{\tilde L}\{d,\{L,d\}\} = 2\{\{L,d\},d\}$. ■

Hodge $*$ operator

Let V be a vector space. **A metric g on V induces a natural metric on each of its tensor spaces:** $g(x_1 \otimes x_2 \otimes ... \otimes x_k, x'_1 \otimes x'_2 \otimes ... \otimes x'_k) = g(x_1, x'_1)g(x_2, x'_2)...g(x_k, x'_k)$.

This gives a natural positive definite scalar product on differential forms over a Riemannian manifold (M, g): $g(\alpha, \beta) := \int_M g(\alpha, \beta) \operatorname{Vol}_M$

Another non-degenerate form is provided by the **Poincare pairing:** $\alpha, \beta \longrightarrow \int_M \alpha \wedge \beta$.

DEFINITION: Let M be a Riemannian n-manifold. Define **the Hodge $*$ operator** $* : \Lambda^k M \longrightarrow \Lambda^{n-k} M$ by the following relation: $g(\alpha, \beta) = \int_M \alpha \wedge * \beta$.

REMARK: The Hodge $*$ operator always exists. It is defined explicitly in an orthonormal basis $\xi_1, ..., \xi_n \in \Lambda^1 M$:

$$*(\xi_{i_1} \wedge \xi_{i_2} \wedge ... \wedge \xi_{i_k}) = (-1)^s \xi_{j_1} \wedge \xi_{j_2} \wedge ... \wedge \xi_{j_{n-k}},$$

where $\xi_{j_1}, \xi_{j_2}, ..., \xi_{j_{n-k}}$ is a complementary set of vectors to $\xi_{i_1}, \xi_{i_2}, ..., \xi_{i_k}$, and s the signature of a permutation $(i_1, ..., i_k, j_1, ..., j_{n-k})$.

REMARK: $*^2\big|_{\Lambda^k(M)} = (-1)^{k(n-k)} \operatorname{Id}_{\Lambda^k(M)}$

$d^* = (-1)^{nk} * d*$

CLAIM: On a compact Riemannian n-manifold, one has $d^*\Big|_{\Lambda^k M} = (-1)^{nk} * d*$, where d^* denotes **the adjoint operator**, which is defined by the equation $(d\alpha, \gamma) = (\alpha, d^*\gamma)$.

Proof: Since

$$0 = \int_M d(\alpha \wedge \beta) = \int_M d(\alpha) \wedge \beta + (-1)^{\tilde\alpha} \alpha \wedge d(\beta),$$

one has $(d\alpha, *\beta) = (-1)^{\tilde\alpha}(\alpha, *d\beta)$. Setting $\gamma := *\beta$, we obtain

$$(d\alpha, \gamma) = (-1)^{\tilde\alpha}(\alpha, *d(*)^{-1}\gamma) = (-1)^{\tilde\alpha}(-1)^{\tilde\alpha(\tilde n - \tilde\alpha)}(\alpha, *d*\gamma) = (-1)^{\tilde\alpha\tilde n}(\alpha, *d*\gamma).$$

■

REMARK: Since in all applications which we consider, n is even, **I would from now on ignore the sign $(-1)^{nk}$.**

Hodge theory

DEFINITION: The anticommutator $\Delta := \{d, d^*\} = dd^* + d^*d$ is called **the Laplacian** of M. It is self-adjoint and positive definite: $(\Delta x, x) = (dx, dx) + (d^*x, d^*x)$. Also, Δ commutes with d and d^* (Lemma 1).

THEOREM: (The main theorem of Hodge theory)
There is a basis in the Hilbert space $L^2(\Lambda^*(M))$ consisting of eigenvectors of Δ.

THEOREM: ("Elliptic regularity for Δ") Let $\alpha \in L^2(\Lambda^k(M))$ be an eigenvector of Δ. **Then α is a smooth k-form.**

These two theorems will be proven in the next lecture.

De Rham cohomology (reminder)

DEFINITION: The space $H^i(M) := \frac{\ker d\big|_{\Lambda^i M}}{d(\Lambda^{i-1} M)}$ is called **the de Rham cohomology of M**.

DEFINITION: A form α is called **harmonic** if $\Delta(\alpha) = 0$.

REMARK: Let α be a harmonic form. **Then** $(\Delta x, x) = (dx, dx) + (d^* x, d^* x)$, hence $\alpha \in \ker d \cap \ker d^*$

REMARK: The projection $\mathcal{H}^i(M) \longrightarrow H^i(M)$ from harmonic forms to cohomology is injective. Indeed, a form α lies in the kernel of such projection if $\alpha = d\beta$, but then $(\alpha, \alpha) = (\alpha, d\beta) = (d^*\alpha, \beta) = 0$.

THEOREM: The natural map $\mathcal{H}^i(M) \longrightarrow H^i(M)$ is an isomorphism (see the next page).

REMARK: Poincare duality immediately follows from this theorem.

Hodge theory and the cohomology (reminder)

THEOREM: The natural map $\mathcal{H}^i(M) \longrightarrow H^i(M)$ is an isomorphism.

Proof. Step 1: Since $d^2 = 0$ and $(d^*)^2 = 0$, one has $\{d, \Delta\} = 0$. This means that **Δ commutes with the de Rham differential.**

Step 2: Consider the eigenspace decomposition $\Lambda^*(M) \hat{=} \bigoplus_\alpha \mathcal{H}^*_\alpha(M)$, where α runs through all eigenvalues of Δ, and $\mathcal{H}^*_\alpha(M)$ is the corresponding eigenspace. **For each α, de Rham differential defines a complex**

$$\mathcal{H}^0_\alpha(M) \xrightarrow{d} \mathcal{H}^1_\alpha(M) \xrightarrow{d} \mathcal{H}^2_\alpha(M) \xrightarrow{d} \ldots$$

Step 3: On $\mathcal{H}^*_\alpha(M)$, one has $dd^* + d^*d = \alpha$. When $\alpha \neq 0$, and η closed, this implies $dd^*(\eta) + d^*d(\eta) = dd^*\eta = \alpha\eta$, hence $\eta = d\xi$, with $\xi := \alpha^{-1}d^*\eta$. This implies that **the complexes $(\mathcal{H}^*_\alpha(M), d)$ don't contribute to cohomology.**

Step 4: We have proven that

$$H^*(\Lambda^* M, d) = \bigoplus_\alpha H^*(\mathcal{H}^*_\alpha(M), d) = H^*(\mathcal{H}^*_0(M), d) = \mathcal{H}^*(M).$$

■

The ring of symbols

THEOREM: Consider the filtration $\operatorname{Diff}^0(M) \subset \operatorname{Diff}^1(M) \subset \operatorname{Diff}^2(M) \subset ...$ on the ring of differential operators. Then **its associated graded ring is isomorphic to the ring $\bigoplus_i \operatorname{Sym}^i(TM)$.**
Proof: Lecture 2. ■

DEFINITION: Let D be a differential operator of order p. Its class in $\operatorname{Diff}^p(M)/\operatorname{Diff}^{p-1}(M)$ is called **symbol** of D. Symbol belongs to $\operatorname{Sym}^p(TM)$. Similarly, for $D \in \operatorname{Diff}^p(F,G)$, symbol is an element of $\operatorname{Diff}^p(F,G)/\operatorname{Diff}^{p-1}(F,G) = \operatorname{Sym}^p(TM) \otimes_{C^\infty M} \operatorname{Hom}(F,G)$.

REMARK: $\operatorname{symb}(AB) = \operatorname{symb}(BA)$. Indeed, **the ring of symbols $\bigoplus_i \operatorname{Diff}^i(M)/\operatorname{Diff}^{i-1}(M)$ is commutative.**

DEFINITION: Let $g \in \operatorname{Sym}^2(T^*M)$ be a Riemannian form. Using g to identify TM and T^*M, we can consider g as an element in $\operatorname{Sym}^2(TM)$. This "Riemannian bivector" is denoted g^{-1}.

We are going to compute the symbol of the Laplacian operator and the "rough Laplacian" $\nabla^*\nabla$. Today we prove the following **"Weitzenböck formula":**

THEOREM: $\operatorname{symb}(\Delta) = \operatorname{symb}(\nabla^*\nabla) = g^{-1} \otimes \operatorname{Id}_{\Lambda^*(M)}$.

Roland Weitzenböck: 26 May 1885 - 24 July 1955

Left to right: Diederik Korteweg, Roland Weitzenböck, Remmelt Sissingh, 1926 in Amsterdam.

...Weitzenböck was elected member of the Royal Netherlands Academy of Arts and Sciences (KNAW) in May 1924, but suspended in May 1945 because of his attitude during the war. Weitzenböck had been a member of the National Socialist Movement in the Netherlands.

In 1923 Weitzenböck published a modern monograph on the theory of invariants on manifolds that included tensor calculus. In the Preface of this monograph one can read an offensive acrostic. One finds that the first letter of the first word in the first 21 sentences spell out:

NIEDER MIT DEN FRANZOSEN

He also published papers on torsion. In fact, in his paper "Differential Invariants in Einstein's Theory of Tele-parallelism" Weitzenböck had given a supposedly complete bibliography of papers on torsion without mentioning Élie Cartan.

Symbol of the connection

CLAIM: Let $d: C^\infty M \longrightarrow \Lambda^1 M$ be the differential. **Then its symbol** $\operatorname{symb}(d) \in TM \otimes \operatorname{Hom}(C^\infty M, \Lambda^1 M)$ **is identity:** $\operatorname{symb}(d) = \operatorname{Id}_{\Lambda^1 M} \in TM \otimes \Lambda^1 M = \operatorname{End}(\Lambda^1 M)$**.**

Proof: $d = \sum_i dx_i \frac{d}{dx_i}$, representing identity in $\Lambda^1 M \otimes TM$. ∎

REMARK: The same is true for the symbol of the connection $\nabla: B \longrightarrow B \otimes \Lambda^1(TM)$:

$$\operatorname{symb}(\nabla) = \operatorname{Id}_{\Lambda^1 M} \otimes \operatorname{Id}_B$$

Indeed, **in local coordinates the connection is written as $\nabla = d + A$,** and A is a differential operator of order 0, hence it does not contribute to symb.

EXERCISE: Let $D: B \longrightarrow B \otimes \Lambda^1(TM)$ be a differential operator with $\operatorname{symb}(D) = \operatorname{symb}(\nabla)$. **Prove that it is a connection.**

Symbol of d and d^*

Claim 1: Let $A: F \longrightarrow G$ be a linear operator, and $D: G \longrightarrow H$ a differential operator. **Then** symb$(AD) = A($symb$(D))$. ■

Claim 2: Let $e: \Lambda^1(M) \otimes \Lambda^*(M) \longrightarrow \Lambda^{*+1}(M)$ be the multiplication operator, and $d: \Lambda^*(M) \longrightarrow \Lambda^{*+1}(M)$ de Rham differential. **Then** symb$(d)(\theta) = e(\theta) \in$ End$(\Lambda^*(M))$ **for any** $\theta \in T^*M$**.** Here we understand symbol as a map from $T^*(M)$ to End$(\Lambda^*(M))$.

Proof: In local coordinates, one has $d = \sum_i e(dx_i)\frac{d}{dx_i}$. ■

DEFINITION: Let i be the **"interior multiplication",**

$$i: \Lambda^1(M) \otimes \Lambda^*(M) \longrightarrow \Lambda^{*-1}(M)$$

with $i(\theta) := (-1)^{nk} * e(\theta) *$. This is an operator which takes a 1-form, uses Riemannian metric to produce a vector field, and takes the convolution with this vector field.

CLAIM: Let $d^* = (-1)^{nk} * d *$. Then symb$(d^*)(\theta) = i(\theta) \in$ End$(\Lambda^*(M))$.

Proof: Follows from Claim 1, Claim 2. ■

Symbol of the Laplacian

CLAIM: Consider a Riemannian manifold (M,g). Let

$$e: \Lambda^1(M)\otimes\Lambda^*(M)\longrightarrow\Lambda^{*+1}(M), \quad i: \Lambda^1(M)\otimes\Lambda^*(M)\longrightarrow\Lambda^{*-1}(M)$$

be the exterior and interior multiplication operators defined above, and $x,y\in\Lambda^1 M$. **Then the anticommutator $\{i_x,e_y\}$ is equal to a multiplication by a function $\tilde{g}(x,y)$**, where $\tilde{g}=g^{-1}$ is the Riemannian form extended to T^*M using the natural isomorphism $T^*M=TM$.

Proof: Let $x_1,...,x_n$ be an orthonormal basis in $\Lambda^1(M)$. Then $\{i_{x_1},e_{x_1}\}$ takes a monomial α without x_1 to $i_{x_1}e_{x_1}\alpha=\alpha$ and takes a monomial $x_1\wedge\alpha$ to $e_{x_1}i_{x_1}(x_1\wedge\alpha)=x_1\wedge\alpha$. Also, i_{x_1} and e_{x_2} anticommute on all monomials. ■

COROLLARY: The symbol of $\Delta=\{d,d^*\}$, evaluated on $x\otimes y$, is equal to $\{i_x,e_y\}=\tilde{g}(x,y)$.

Proof: Symbol is multiplicative: symb(A) symb$(B)=$ symb(AB). The symbol of d is e, and the symbol of d^* is i. This gives

$$\operatorname{symb}(\Delta)(x\otimes y)=\{\operatorname{symb}(d),\operatorname{symb}(d^*)\}(x\otimes y)=\{e_x,i_y\}=\tilde{g}(x,y).$$

■

Symbol of the rough Laplacian

CLAIM: Consider a Riemannian manifold (M, g), and let ∇ be a connection on a bundle B. **Then** $\operatorname{symb}(\nabla^*\nabla)$**, evaluated on** $x \otimes y \in T^*(M) \otimes T^*(M)$**, is equal to** $\tilde{g}(x, y)\operatorname{Id}_B$, where $\tilde{g} = g^{-1}$ is the Riemannian form extended to T^*M using the natural isomorphism $T^*M = TM$.

Proof: The symbol of ∇ takes $x \in T^*(M)$ to $b \longrightarrow b \otimes x$, and the symbol of $\nabla^* : B \otimes \Lambda^1(M) \longrightarrow B$ takes $x \in T^*(M)$ to an operator $b \otimes y \longrightarrow \tilde{g}(x, y)b$. Using $\operatorname{symb}(A)\operatorname{symb}(B) = \operatorname{symb}(AB)$, we obtain that $\operatorname{symb}(\nabla^*\nabla)$ evaluated on $x \otimes y$ is equal to the multiplication by $\tilde{g}(x, y)$. ■