Hodge theory

lecture 8: Sobolev lemma

NRU HSE, Moscow

Misha Verbitsky, February 17, 2018

Fredholm operators (reminder)

DEFINITION: A continuous operator $F: H_1 \longrightarrow H_2$ of Hilbert spaces is called **Fredholm** if its image is closed and kernel and cokernel are finite-dimensional.

REMARK: "Cokernel" of a morphism $F: H_1 \longrightarrow H_2$ of topological vector spaces is often defined as $\frac{H_2}{\overline{\operatorname{im} F}}$.

DEFINITION: An operator $F: H_1 \longrightarrow H_2$ **has finite rank** if its image has finite rank.

CLAIM: An operator $F: H_1 \longrightarrow H_2$ **is Fredholm** if and only if there exists $F_1: H_2 \longrightarrow H_1$ such that **the operators $\operatorname{Id} - FF_1$ and $\operatorname{Id} - F_1F$ have finite rank.**

Proof: This is because F defines an isomorphism $F: H_1 / \ker F \longrightarrow \operatorname{im} F$ as shown above. ■

Connections (reminder)

DEFINITION: Recall that **a connection** on a bundle B is an operator $\nabla : B \longrightarrow B \otimes \Lambda^1 M$ satisfying $\nabla(fb) = b \otimes df + f\nabla(b)$, where $f \longrightarrow df$ is de Rham differential. When X is a vector field, we denote by $\nabla_X(b) \in B$ the term $\langle \nabla(b), X \rangle$.

REMARK: In local coordinates, connection on B is a sum of differential and a form $A \in \operatorname{End} B \otimes \Lambda^1 M$. Therefore, ∇_X is a derivation along X plus linear endomorphism. This implies that **any first order differential operator on B is expressed as a linear combination of the compositions of covariant derivatives ∇_X and linear maps.**

This follows from the definition of the first order differential operator: **by definition, it is a linear combination of partial derivatives combined with linear maps.**

L^2_p-metrics and connections (reminder)

DEFINITION: Let F be a vector bundle on a compact manifold. The **L^2_p-topology** on the space of sections of F is a topology defined by the norm $|f|_p$ with $|f|_p^2 = \sum_{i=0}^p \int_M |\nabla^i f|^2 \operatorname{Vol}_M$, for some connection and scalar product on F and $\Lambda^1 M$.

REMARK: **The metric $|f|_p^2$ is equivalent to the Sobolev's L^2_p-metric on $C^\infty(M)$.** Indeed, all partial derivatives of a function f are expressed through $\nabla^i f$, hence an L^2-bound on partial derivatives gives L^2-bound on $\nabla^i f$, and is given by such a bound.

From now on, **we write (x,y) instead of $\int_M (x,y) \operatorname{Vol}_M$.** This metric is also denoted L^2; the space of sections of B with this metric (B, L^2).

DEFINITION: We define the **Sobolev's L^2_p-metric on vector bundles** by $L^2_p(x,y) = \sum_{i=0}^p (\nabla^i(x), \nabla^i(y))$.

Properties of L^2_p-metric

These results were proven earlier.

CLAIM: The Sobolev's L^2_p-metric is equivalent to $g(x,y) := (\nabla^p(x), \nabla^p(y)) + (x,y)$.

THEOREM: (Rellich lemma) Let M be a compact manifold. **Then the identity map $L^2_p(M) \longrightarrow L^2_{p-1}(M)$ is compact.**

THEOREM: Let B be a vector bundle, and $D: B \longrightarrow B$ a differential operator which has the same symbol as $(\nabla^p)^* \nabla^p$. **Then $D: (B, L^2_{2p}) \longrightarrow (B, L^2)$ is Fredholm.**

Laplace operators (reminder)

DEFINITION: Let M be a Riemannian manifold, and $d: \Lambda^*(M) \longrightarrow \Lambda^{*+1}(M)$ de Rham differential. Then $\Delta := dd^* + d^*d$ is called **the Laplacian**.

DEFINITION: Let M be a Riemannian manifold, and B a bundle with orthogonal metric and a connection $\nabla: B \longrightarrow B \otimes \Lambda^1 M$. Using the formula $\nabla(b \otimes \eta) = \nabla(b) \wedge \eta + b \otimes d\eta$, we extend ∇ to an operator $\nabla: B \otimes \Lambda^i M \longrightarrow B \otimes \Lambda^{i+1} M$ satisfying the Leibnitz equation. This operator is denoted d_∇ to distinguish it from the connection. **The Laplacian with coefficients in B** is $d_\nabla d_\nabla^* + d_\nabla^* d_\nabla$.

THEOREM: The Laplacian has the same symbol $\sigma \in \mathrm{Sym}^2(TM) \otimes \mathrm{End}(\Lambda^* M \otimes B)$ as $\nabla^*\nabla$, and it is equal to $g^{-1} \otimes \mathrm{Id}_{B \otimes \Lambda^* M}$, where $g^{-1} \in \mathrm{Sym}^2 TM$ is the bivector which corresponds to the Riemannian metric.

The following corollary is immediate.

COROLLARY: The Laplacian is a Fredholm map from $(\Lambda^*(M) \otimes B, L^2_p)$ to $(\Lambda^*(M) \otimes B, L^2_{p-2})$.

Proof: Indeed, **Laplacian is a sum of a Fredholm map $(\nabla^*)\nabla$ and a compact operator** (all lower order differential operators are compact by Rellich lemma). ■

Green operator

CLAIM: Let $\Delta : (\Lambda^*(M), L^2_2) \longrightarrow (\Lambda^*(M), L^2)$ be the Laplacian operator **Then** $\operatorname{im}\Delta = \ker\Delta^\perp$**, taken with respect to the** L^2**-metric.**

Proof: Since Δ is self-adjoint with respect to L^2-metric, for each $x, y \in (\Lambda^*(M), L^2_2)$ one has $(x, \Delta y) = 0 \Leftrightarrow (\Delta x, y) = 0$. Therefore, $x \in \ker\Delta \Leftrightarrow x \perp \operatorname{im}\Delta$. ■

COROLLARY: The restriction $\Delta : (\operatorname{im}\Delta, L^2_2) \longrightarrow (\operatorname{im}\Delta, L^2)$ **is an isomorphism of Hilbert spaces.** ■

DEFINITION: The Green operator G_Δ is a map $(\Lambda^*(M), L^2) \xrightarrow{G_\Delta} (\Lambda^*(M), L^2)$ defined as Δ^{-1} on $\operatorname{im}\Delta$ and as 0 on $\ker\Delta = \operatorname{im}\Delta^\perp$.

CLAIM: The Green operator $G_\Delta : (\Lambda^*(M), L^2) \longrightarrow (\Lambda^*(M), L^2)$ **is self-adjoint and compact** in the usual L^2-metric on $\Lambda^*(M)$.

Proof: Since Δ is self-adjoint on $\operatorname{im}\Delta$, the same is true for Δ^{-1}. However, when $x \in \operatorname{im}\Delta^\perp$, one has $0 = (G_\Delta x, y) = (x, G_\Delta y)$ as shown above. Compactness follows immediately from Rellich lemma, because G_Δ is a composition of a continuous operator $(\Lambda^*(M), L^2) \xrightarrow{\Delta^{-1}} (\Lambda^*(M), L^2_2)$ and a compact map $(\Lambda^*(M), L^2_2) \xrightarrow{\operatorname{Id}} (\Lambda^*(M), L^2)$. ■

Green operator diagonalizes

THEOREM: The Green operator $G_\Delta : (\Lambda^*(M), L^2) \longrightarrow (\Lambda^*(M), L^2)$ **can be diagonalized in an orthonormal basis.** Its eigenvalues are non-negative and converge to 0, and each eigenspace is finite-dimensional.

Proof: Follows from the von Neumann spectral theorem. ■

Today I will prove the following theorem.

THEOREM: Let $\alpha \in L^2(\Lambda^*(M))$ be an eigenvector of G_Δ, $G_\Delta(\alpha) = \lambda\alpha$. **Then α is smooth.**

Proof: Notice that the identiry map $L^2_p(\Lambda^*(M)) \longrightarrow L^2_{p-i}(\Lambda^*(M))$ is continuous for all $i \geqslant 0$. This gives a natural chain of embeddings $L^2_p(\Lambda^*(M)) \subset L^2_{p-1}(\Lambda^*(M)) \subset ... \subset L^2(\Lambda^*(M))$. Since $\lambda^k\alpha = G^k_\Delta(\alpha)$ belongs to $L^2_{2k}(\Lambda^*(M))$, we have $\alpha \in \bigcap_p L^2_p(\Lambda^*(M))$. Then the theorem is implied by the following result of Sobolev, proven later today.

THEOREM: (Sobolev) Any vector in the intersection of all $L^2_p(\Lambda^*(M))$ is represented by a smooth form: $\bigcap_p L^2_p(\Lambda^*(M)) = \Lambda^*(M)$.

REMARK: The same arguments work for Laplacian with coefficients in a vector bundle.

Sobolev lemma

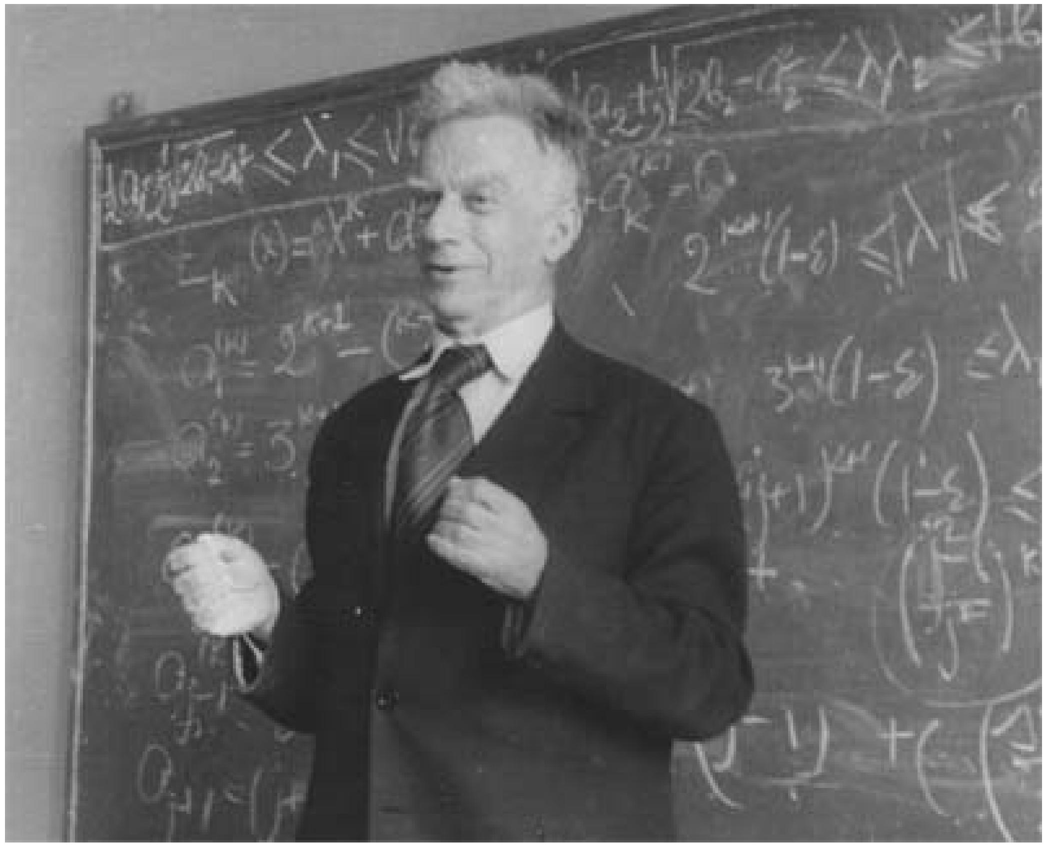
DEFINITION: Let B be a bundle over M. Recall that **C^l-topology** on the space of sections $C^l(B)$ of B of class C^l is defined by the norm $|b|_{C^p} = \sup_M \sum_{i=0}^l |\nabla^i b|$.

EXERCISE: Prove that $C^l(B)$ **is a Banach space with respect to this norm.**

Sobolev's theorem $\bigcap_p L^2_p(\Lambda^*(M)) = \Lambda^*(M)$ is immediately implied by the following lemma.

THEOREM: (Sobolev lemma)
Let $\{b_i\}$ be a sequence of sections of a vector bundle B over a manifold M with $\dim M = n$, converging to b in L^2_s, where $s > l + \frac{n}{2}$. **Then it converges to a section in $C^l(B)$ in C^l-topology.**

It is proven later today.



Sergei Lvovich Sobolev
6 October 1908 - 3 January 1989

Fourier series (reminder)

CLAIM: ("Fourier series") Functions $e_k(t) = e^{2\pi\sqrt{-1}\,kt}$, $k \in \mathbb{Z}$ on $S^1 = \mathbb{R}/\mathbb{Z}$ **form an orthonormal basis in the space $L^2(S^1)$** of square-integrable functions on the circle.

Proof: Orthogonality is clear from $\int_{S^1} e^{2\pi\sqrt{-1}\,kt} dt = 0$ for all $k \neq 0$ (prove it). To show that the space of Fourier polynomials $\sum_{i=-n}^{n} a_k e_k(t)$ is dense in the space of continuous functions on circle, use the Stone-Weierstrass approximation theorem, applied to the ring $R = \langle \sin(mx), \cos(nx) \rangle$ of functions obtained from real and imaginary parts of $e^{2\pi\sqrt{-1}\,kt}$. ■

DEFINITION: Fourier monomials on a torus are functions $F_{l_1,...,l_n} := \exp(2\pi\sqrt{-1}\sum_{i=1}^{n} l_i t_i)$, where $l_1, ..., l_n \in \mathbb{Z}$.

CLAIM: Fourier monomials **form an orthonormal basis in the space $L^2(T^n)$** of square-integrable functions on the torus T^n.

Proof: The same. ■

Sobolev's L^2-norm on $C_c^\infty(\mathbb{R}^n)$ (reminder)

DEFINITION: Denote by $C_c^\infty(\mathbb{R}^n)$ the space of smooth functions with compact support. For each differential monomial

$$P_\alpha = \frac{\partial^{k_1}}{\partial x_1^{k_1}} \frac{\partial^{k_2}}{\partial x_2^{k_2}} ... \frac{\partial^{k_n}}{\partial x_1^{k_n}}$$

consider the corresponding partial derivative

$$P_\alpha(f) = \frac{\partial^{k_1}}{\partial x_1^{k_1}} \frac{\partial^{k_2}}{\partial x_2^{k_2}} ... \frac{\partial^{k_n}}{\partial x_1^{k_n}} f.$$

Given $f \in C_c^\infty(\mathbb{R}^n)$, one defines **the L_p^2 Sobolev's norm $|f|_p$** as follows:

$$|f|_s^2 = \sum_{\deg P_\alpha \leqslant p} \int |P_\alpha(f)|^2 \operatorname{Vol}$$

where the sum is taken over all differential monomials P_α of degree $\leqslant p$, and $\operatorname{Vol} = dx_1 \wedge dx_2 \wedge ... dx_n$ - the standard volume form.

REMARK: Same formula defines **Sobolev's L^2-norm L_p^2 on the space of smooth functions on a torus T^n.**

Sobolev's L^2-norm on a torus (reminder)

CLAIM: The Fourier monomials $F_{l_1,...,l_n} := e^{2\pi\sqrt{-1}\,\sum l_i t_i}$ **are eigenvectors for the differential monomials** $P_\alpha = \frac{\partial^{k_1}}{\partial x_1^{k_1}}\frac{\partial^{k_2}}{\partial x_2^{k_2}}...\frac{\partial^{k_n}}{\partial x_1^{k_n}}$**. Moreover,** $P_\alpha(F_{l_1,...,l_n}) = \prod_{i=1}^n (2\pi\sqrt{-1}\,k_i)^{l_i}$. ■

COROLLARY: The Fourier monomials are orthogonal in the Sobolev's L^2_p-metric, and

$$|F_{l_1,...,l_n}|^2_{2,p} = \sum_{k_1+...+k_n=1}^{p} \prod_{i=1}^{n} (2\pi l_i)^{2k_i}.$$

■

THEOREM: (Rellich lemma for a torus)
The identity map $L^2_p(T^n) \longrightarrow L^2_{p-1}(T^n)$**. is compact.**

Sobolev Lemma on a circle

LEMMA: Consider Fourier series on S^1: $f(t) := \sum_{k\in\mathbb{Z}} \tau_k e^{2\pi\sqrt{-1}\, kt}$. Suppose that $\sum_{k\in\mathbb{Z}} k^{2+l}|\tau_k|^2$ converges. **Then $\sum_{k\in\mathbb{Z}} \tau_k e^{2\pi\sqrt{-1}\, kt}$ converges to a function of class C^l in C^l-topology.**

Proof. Step 1: If $l=0$, convergence of $\sum_{k\in\mathbb{Z}} k^2|\tau_k|^2$ **implies that $\sum_{k\in\mathbb{Z}} \tau_k e^{2\pi\sqrt{-1}\, kt}$ converges absolutely,** because the Cauchy-Schwarz inequality $(\sum a_i b_i)^2 \leqslant \sum a_i^2 \sum b_i^2$ gives after putting $a_i b_i = |\tau_i|$, $a_i = i|\tau_i|$

$$\sum_k k^2|\tau_k|^2 \geqslant \left(\sum_k |\tau_k|\right)^2 \left(\sum_{k=0}^{\infty} k^{-2}\right)^{-2}$$

Therefore it converges in C^0-topology.

Step 2: $\frac{d^k f}{dt^k} = \sum_{k\in\mathbb{Z}} k^l \tau_k e^{2\pi\sqrt{-1}\, kt}$, and this series converges absolutely when $\sum_{k\in\mathbb{Z}} k^{2+l}|\tau_k|^2 < \infty$ for the same reason. ■

COROLLARY: Let $\{f_i\}$ be a sequence of smooth functions on S^1 which converges in L^2_{p+1}. **Then it also converges in C^p-topology.** ■

Sobolev Lemma on a torus

LEMMA: Consider Fourier series on a torus T^n:

$$f = \sum_{k_1,\dots k_n \in \mathbb{Z}^n} \tau_{k_1,\dots k_n} e^{2\pi\sqrt{-1}\,\sum_{i=1}^n k_i t_i} \quad (*)$$

Suppose that

$$\sum_{k_1,\dots k_n \in \mathbb{Z}^n} |\tau_{k_1,\dots k_n}|^2 \sum_{i=1}^n k_i^{2+2n+l} \quad (**)$$

converges. **Then $\sum_{k\in\mathbb{Z}} \tau_k e^{2\pi\sqrt{-1}\,kt}$ converges to a function in C^l-topology.**

Proof. Step 1: If $l=0$, convergence of (**) implies absolute convergence of (*). Indeed, the Cauchy-Schwarz inequality $(\sum a_\alpha b_\alpha)^2 \leqslant \sum a_\alpha^2 \sum b_\alpha^2$ applied to $a_\alpha b_\alpha = |\tau_\alpha|$, $a_\alpha = \sum k_i^{2+2n}|\tau_\alpha|$, where $\alpha = (k_1, ..., k_n)$ is a multi-index, gives

$$\sum_{k_1,\dots k_n} \sum k_i^{2+2n} |\tau_{k_1,\dots k_n}|^2 \geqslant \left(\sum_{k_1,\dots k_n} |\tau_{k_1,\dots k_n}| \right)^2 \left(\sum_{k_1,\dots k_n}^n k_i^{-2-2n} \right)^{-2}.$$

The last sum converges, hence $\sum |\tau_{k_1,\dots k_n}|$ converges in C^0.

Step 2: Same computation as above (left as an exercise). ■

Sobolev Lemma

COROLLARY: Let $\{f_i\}$ be a sequence of smooth functions on a torus T^n which converges in L^2_s, with $s > l + \frac{n}{2}$. **Then it also converges in C^l-topology.** ■

THEOREM: (Sobolev lemma) Let B be a bundle on a compact manifold M, and $\{f_i\}$ be a sequence of smooth functions which converges in L^2_s, with $s > l + \frac{n}{2}$. **Then it also converges in C^l-topology.**

Proof. Step 1: Let $\{U_j\}$ be a finite atlas on M and $\{\psi_j\}$ the corresponding partition of unity. We will identify U_j with bounded subsets in $\mathbb{R}^n$. Then $|f|^2_p = \sum_j |\psi_j f|^2_p$, where the second $|\cdot|^2_p$-norm is taken on a bounded subset in $\mathbb{R}^n$, considered as a subset in a torus.

Step 2: Let $\{f_i\}$ be a sequence of sections of a bundle B converging to f in L^2_s. Then $\{\psi_j f_i\}$ converges to $\psi_j f$ in L^2_s. Applying Sobolev lemma for torus, we obtain that $\{\psi_j f_i\}$ converges to $\psi_j f$ in C^l. ■