

Hodge theory

lecture 9: Complex manifolds

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Complex structure on vector spaces

DEFINITION: Let V be a vector space over \mathbb{R} , and $I : V \rightarrow V$ an automorphism which satisfies $I^2 = -\text{Id}_V$. Such an automorphism is called **a complex structure operator** on V .

We extend the action of I on the tensor spaces $V \otimes V \otimes \dots \otimes V \otimes V^* \otimes V^* \otimes \dots \otimes V^*$ by multiplicativity: $I(v_1 \otimes \dots \otimes w_1 \otimes \dots \otimes w_n) = I(v_1) \otimes \dots \otimes I(w_1) \otimes \dots \otimes I(w_n)$.

Trivial observations:

1. **The eigenvalues α_i of I are $\pm\sqrt{-1}$.** Indeed, $\alpha_i^2 = -1$.
2. **V admits an I -invariant, positive definite scalar product (“metric”)** g . Take any metric g_0 , and let $g := g_0 + I(g_0)$.
3. **I is orthogonal for such g .**
Indeed, $g(Ix, Iy) = g_0(x, y) + g_0(Ix, Iy) = g(x, y)$.
4. **I diagonalizable over \mathbb{C} .** Indeed, any orthogonal matrix is diagonalizable.
5. **There are as many $\sqrt{-1}$ -eigenvalues as there are $-\sqrt{-1}$ -eigenvalues.**

Hermitian structures

DEFINITION: An I -invariant positive definite scalar product on (V, I) is called **an Hermitian metric**, and (V, I, g) – an Hermitian space.

REMARK: Let I be a complex structure operator on a real vector space V , and g – a Hermitian metric. Then **the bilinear form** $\omega(x, y) := g(x, Iy)$ **is skew-symmetric**. Indeed, $\omega(x, y) = g(x, Iy) = g(Ix, I^2y) = -g(Ix, y) = -\omega(y, x)$.

DEFINITION: A skew-symmetric form $\omega(x, y)$ is called **an Hermitian form on** (V, I) .

REMARK: In the triple I, g, ω , **each element can recovered from the other two**.

The Grassmann algebra

DEFINITION: Let V be a vector space. Denote by $\Lambda^i V$ the space of antisymmetric polylinear i -forms on V^* , and let $\Lambda^* V := \bigoplus \Lambda^i V$. Denote by $T^{\otimes i} V$ the algebra of **all** polylinear i -forms on V^* (“tensor algebra”), and let $\text{Alt} : T^{\otimes i} V \rightarrow \Lambda^i V$ be **the antisymmetrization**,

$$\text{Alt}(\eta)(x_1, \dots, x_i) := \frac{1}{i!} \sum_{\sigma \in \Sigma_i} (-1)^{\tilde{\sigma}} \eta(x_{\sigma_1}, \dots, x_{\sigma_i})$$

where Σ_i is the group of permutations, and $\tilde{\sigma} = 1$ for odd permutations, and 0 for even. Consider the multiplicative operation (“wedge-product”) on $\Lambda^* V$, denoted by $\eta \wedge \nu := \text{Alt}(\eta \otimes \nu)$. The space $\Lambda^* V$ with this operation is called **the Grassmann algebra**.

REMARK: It is an algebra of anti-commutative polynomials.

Properties of Grassmann algebra:

1. $\dim \Lambda^i V := \binom{\dim V}{i}$, $\dim \Lambda^* V = 2^{\dim V}$.
2. $\Lambda^*(V \oplus W) = \Lambda^*(V) \otimes \Lambda^*(W)$.

The Hodge decomposition in linear algebra

DEFINITION: Let (V, I) be a space equipped with a complex structure. **The Hodge decomposition** $V \otimes_{\mathbb{R}} \mathbb{C} := V^{1,0} \oplus V^{0,1}$ is defined in such a way that $V^{1,0}$ is a $\sqrt{-1}$ -eigenspace of I , and $V^{0,1}$ a $-\sqrt{-1}$ -eigenspace.

REMARK: Let $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. The Grassmann algebra of skew-symmetric forms $\Lambda^n V_{\mathbb{C}} := \Lambda_{\mathbb{R}}^n V \otimes_{\mathbb{R}} \mathbb{C}$ admits a decomposition

$$\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}$$

We denote $\Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}$ by $\Lambda^{p,q} V$. The resulting decomposition $\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^{p,q} V$ is called **the Hodge decomposition of the Grassmann algebra**.

$U(1)$ -representations and the weight decomposition

REMARK: The operator I induces $U(1)$ -action on V by the formula $\rho(t)(v) = \cos t \cdot v + \sin t \cdot I(v)$. We extend this action on the tensor spaces by multiplicativity.

REMARK: Any complex representation W of $U(1)$ is written as a sum of 1-dimensional representations $W_i(p)$, with $U(1)$ acting on each $W_i(p)$ as $\rho(t)(v) = e^{\sqrt{-1}pt}(v)$. The 1-dimensional representations are called **weight p representations of $U(1)$** .

DEFINITION: A **weight decomposition** of a $U(1)$ -representation W is a decomposition $W = \bigoplus W^p$, where each $W^p = \bigoplus_i W_i(p)$ is a sum of 1-dimensional representations of weight p .

REMARK: The Hodge decomposition $\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^{p,q} V$ is a **weight decomposition**, with $\Lambda^{p,q} V$ being a weight $p - q$ -component of $\Lambda^n V_{\mathbb{C}}$.

REMARK: $V^{p,p}$ is the space of $U(1)$ -invariant vectors in $\Lambda^{2p} V$.

Further on, TM is the tangent bundle on a manifold, and $\Lambda^i M$ the space of differential i -forms. It is a Grassmann algebra on TM .

Holomorphic functions

DEFINITION: Let $I : TM \rightarrow TM$ be an endomorphism of a tangent bundle satisfying $I^2 = -\text{Id}$. Then I is called **almost complex structure operator**, and the pair (M, I) **an almost complex manifold**.

EXAMPLE: $M = \mathbb{C}^n$, with complex coordinates $z_i = x_i + \sqrt{-1} y_i$, and $I(d/dx_i) = d/dy_i$, $I(d/dy_i) = -d/dx_i$.

EXAMPLE: In complex dimension 1, **almost complex structure is the same as conformal structure with orientation (prove it)**.

DEFINITION: A function $f : M \rightarrow \mathbb{C}$ on an almost complex manifold is called **holomorphic** if $df \in \Lambda^{1,0}(M)$.

REMARK: For some almost complex manifolds, **there are no holomorphic functions at all**, even locally.

Example: S^6 with a certain canonical (G_2 -invariant) almost complex structure.

Holomorphic functions on \mathbb{C}^n

THEOREM: Let $f : M \rightarrow \mathbb{C}$ be a differentiable function on an open subset $M \subset \mathbb{C}^n$, with the natural almost complex structure. **Then the following are equivalent.**

- (1) f is holomorphic.
- (2) The differential $df : TM \rightarrow \mathbb{C}$, considered as a form on the vector space $T_x M = T_x \mathbb{C}^n = \mathbb{C}^n$ is \mathbb{C} -linear.
- (3) For any complex affine line $L \subset \mathbb{C}^n$, the restriction $f|_L : L \rightarrow \mathbb{C}$ is holomorphic (complex analytic) as a function of one complex variable.
- (4) f is expressed as a sum of Taylor series around any point $(z_1, \dots, z_n) \in M$:

$$f(z_1 + t_1, z_2 + t_2, \dots, z_n + t_n) = \sum_{i_1, \dots, i_n} a_{i_1, \dots, i_n} t_1^{i_1} t_2^{i_2} \dots t_n^{i_n}.$$

(here we assume that the complex numbers t_i satisfy $|t_i| < \varepsilon$, where ε depends on f and M).

Proof: (1) and (2) are tautologically equivalent. Equivalence of (1) and (3) is also clear, because a restriction of $\theta \in \Lambda^{1,0}(M)$ to a line is a $(1,0)$ -form on a line, and, conversely, if df is of type $(1,0)$ on each complex line, it is of type $(1,0)$ on TM , which is implied by the following linear-algebraic observation.

Holomorphic functions on \mathbb{C}^n (2)

THEOREM: Let $f : M \rightarrow \mathbb{C}$ be a differentiable function on an open subset $M \subset \mathbb{C}^n$, with the natural almost complex structure. **Then the following are equivalent.**

- (1) f is holomorphic.
- (2) The differential $df : TM \rightarrow \mathbb{C}$, considered as a form on the vector space $T_x M = T_x \mathbb{C}^n = \mathbb{C}^n$ is \mathbb{C} -linear.
- (3) For any complex affine line $L \subset \mathbb{C}^n$, the restriction $f|_L : L \rightarrow \mathbb{C}$ is holomorphic (complex analytic) as a function of one complex variable.
- (4) f is expressed as a sum of Taylor series around any point $(z_1, \dots, z_n) \in M$.

LEMMA: Let $\eta \in V^* \otimes \mathbb{C}$ be a complex-valued linear form on a vector space (V, I) equipped with a complex structure. **Then $\eta \in \Lambda^{1,0}(V)$ if and only if its restriction to any I -invariant 2-dimensional subspace L belongs to $\Lambda^{1,0}(L)$.**

EXERCISE: Prove it.

(4) clearly implies (2). (1) implies (4) by Cauchy formula.

Taylor decomposition from Cauchy formula

Taylor series decomposition on a line is implied by the Cauchy formula:

$$\int_{\partial\Delta} \frac{f(z)dz}{z-a} = 2\pi\sqrt{-1} f(a),$$

where $\Delta \subset \mathbb{C}$ is a disk, $a \in \Delta$ any point, and z coordinate on \mathbb{C} . Indeed, in this case,

$$2\pi\sqrt{-1} f(a) = \sum_{i \geq 0} a^i \int_{\partial\Delta} f(z)(z^{-1})^{i+1},$$

because $\frac{1}{z-a} = z^{-1} \sum_{i \geq 0} (az^{-1})^i$.

Cauchy formula

Let's prove Cauchy formula, using Stokes' theorem. Since the space $\Lambda^{1,0}\mathbb{C}$ is 1-dimensional, $df \wedge dz = 0$ for any holomorphic function on \mathbb{C} . This gives

CLAIM: A function on a disk $\Delta \subset \mathbb{C}$ **is holomorphic if and only if the form $\eta := f dz$ is closed** (that is, satisfies $d\eta = 0$). ■

Now, let S_ε be a radius ε circle around a point $a \in \Delta$, Δ_ε its interior, and $\Delta_0 := \Delta \setminus \Delta_\varepsilon$. Stokes' theorem gives

$$0 = \int_{\Delta_0} d\left(\frac{f(z)dz}{z-a}\right) = - \int_{S_\varepsilon} \frac{f(z)dz}{z-a} + \int_{\partial\Delta} \frac{f(z)dz}{z-a},$$

hence Cauchy formula would follow if we show that $\lim_{\varepsilon \rightarrow 0} \int_{S_\varepsilon} \frac{f(z)dz}{z-a} = 2\pi\sqrt{-1}f(a)$.

Assuming for simplicity $a = 0$ and parametrizing the circle S_ε by $\varepsilon e^{\sqrt{-1}t}$, we obtain

$$\begin{aligned} \int_{S_\varepsilon} \frac{f(z)dz}{z} &= \int_0^{2\pi} \frac{f(\varepsilon e^{\sqrt{-1}t})}{\varepsilon e^{\sqrt{-1}t}} d(\varepsilon e^{\sqrt{-1}t}) = \\ &= \int_0^{2\pi} \frac{f(\varepsilon e^{\sqrt{-1}t})}{\varepsilon e^{\sqrt{-1}t}} \sqrt{-1} \varepsilon e^{\sqrt{-1}t} dt = \int_0^{2\pi} f(\varepsilon e^{\sqrt{-1}t}) \sqrt{-1} dt \end{aligned}$$

as ε tends to 0, $f(\varepsilon e^{\sqrt{-1}t})$ tends to $f(0)$, and this integral goes to $2\pi\sqrt{-1}f(0)$.

Sheaves

DEFINITION: A **presheaf of functions** on a topological space M is a collection of subrings $\mathcal{F}(U) \subset C(U)$ in the ring $C(U)$ of all functions on U , for each open subset $U \subset M$, such that the restriction of every $\gamma \in \mathcal{F}(U)$ to an open subset $U_1 \subset U$ belongs to $\mathcal{F}(U_1)$.

DEFINITION: A presheaf of functions \mathcal{F} is called **a sheaf of functions** if these subrings satisfy the following condition. Let $\{U_i\}$ be a cover of an open subset $U \subset M$ (possibly infinite) and $f_i \in \mathcal{F}(U_i)$ a family of functions defined on the open sets of the cover and compatible on the pairwise intersections:

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$

for every pair of members of the cover. **Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i .**

Sheaves and presheaves: examples

Examples of sheaves:

- * Space of continuous functions
- * Space of smooth functions, any differentiability class
- * Space of real analytic functions

Examples of presheaves which are not sheaves:

- * Space of constant functions (why?)
- * Space of bounded functions (why?)

Ringed spaces

A **ringed space** (M, \mathcal{F}) is a topological space equipped with a sheaf of functions. A **morphism** $(M, \mathcal{F}) \xrightarrow{\Psi} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \xrightarrow{\Psi} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $\psi^* f := f \circ \Psi$ belongs to the ring $\mathcal{F}(\Psi^{-1}(U))$. An **isomorphism** of ringed spaces is a homeomorphism Ψ such that Ψ and Ψ^{-1} are morphisms of ringed spaces.

EXAMPLE: Let M be a manifold of class C^i and let $C^i(U)$ be the space of functions of this class. **Then C^i is a sheaf of functions, and (M, C^i) is a ringed space.**

REMARK: Let $f : X \rightarrow Y$ be a smooth map of smooth manifolds. Since a pullback $f^* \mu$ of a smooth function $\mu \in C^\infty(M)$ is smooth, **a smooth map of smooth manifolds defines a morphism of ringed spaces.**

Complex manifolds

DEFINITION: A holomorphic function on \mathbb{C}^n is a function $f : \mathbb{C}^n \rightarrow \mathbb{C}$ such that df is complex linear, that is $df \in \Lambda^{1,0}(M)$.

REMARK: Holomorphic functions form a sheaf.

DEFINITION: A complex manifold M is a ringed space which is locally isomorphic to an open ball in \mathbb{C}^n with a sheaf of holomorphic functions.

REMARK: In other words, M is covered with open balls embedded to \mathbb{C}^n and transition functions (being coordinate functions for one ball considered in other coordinate system) are holomorphic.

Complex manifolds and almost complex manifolds

DEFINITION: Standard almost complex structure is $I(d/dx_i) = d/dy_i$, $I(d/dy_i) = -d/dx_i$ on \mathbb{C}^n with complex coordinates $z_i = x_i + \sqrt{-1} y_i$.

DEFINITION: A map $\Psi : (M, I) \rightarrow (N, J)$ from an almost complex manifold to an almost complex manifold is called **holomorphic** if $\Psi^*(\Lambda^{1,0}(N)) \subset \Lambda^{1,0}(M)$.

REMARK: This is the same as $d\Psi$ being complex linear; for standard almost complex structures, **this is the same as the coordinate components of Ψ being holomorphic functions.**

DEFINITION: A complex manifold is a manifold equipped with an atlas with charts identified with open subsets of \mathbb{C}^n and transition functions holomorphic.

Integrability of almost complex structures

DEFINITION: An almost complex structure I on a manifold is called **integrable** if any point of M has a neighbourhood U diffeomorphic to an open subset of \mathbb{C}^n , in such a way that the almost complex structure I is induced by the standard one on $U \subset \mathbb{C}^n$.

CLAIM: Complex structure on a manifold M uniquely determines an integrable almost complex structure, and is determined by it.

Proof: Complex structure on a manifold M is determined by the sheaf of holomorphic functions \mathcal{O}_M , and \mathcal{O}_M is determined by I as explained above. ■

Frobenius form

CLAIM: Let $B \subset TM$ be a sub-bundle of a tangent bundle of a smooth manifold. Given vector fields $X, Y \in B$, consider their commutator $[X, Y]$, and let $\psi(X, Y) \in TM/B$ be the projection of $[X, Y]$ to TM/B . **Then $\psi(X, Y)$ is $C^\infty(M)$ -linear in X, Y :**

$$\psi(fX, Y) = \psi(X, fY) = f\psi(X, Y).$$

Proof: Leibnitz identity gives $[X, fY] = f[X, Y] + X(f)Y$, and the second term belongs to B , hence does not influence the projection to TM/B . ■

DEFINITION: This form is called **the Frobenius form** of the sub-bundle $B \subset TM$. This bundle is called **involutive**, or **integrable**, or **holonomic** if $\psi = 0$.

EXERCISE: Give an example of a non-integrable sub-bundle.

Formal integrability

DEFINITION: An almost complex structure I on (M, I) is called **formally integrable** if $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$, that is, if $T^{1,0}M$ is involutive.

DEFINITION: The Frobenius form $\Psi \in \Lambda^{2,0}M \otimes TM$ is called **the Nijenhuis tensor**.

CLAIM: If a complex structure I on M is integrable, it is formally integrable.

Proof: Locally, the bundle $T^{1,0}(M)$ is generated by d/dz_i , where z_i are complex coordinates. These vector fields commute, hence satisfy $[d/dz_i, d/dz_j] \in T^{1,0}(M)$. This means that the Frobenius form vanishes. ■

THEOREM: (Newlander-Nirenberg)

A complex structure I on M is integrable if and only if it is formally integrable.

Proof: (real analytic case) next lecture.

REMARK: In dimension 1, formal integrability is automatic. Indeed, $T^{1,0}M$ is 1-dimensional, hence all skew-symmetric 2-forms on $T^{1,0}M$ vanish.

Distributions

DEFINITION: Distribution on a manifold is a sub-bundle $B \subset TM$

REMARK: Let $\Pi : TM \rightarrow TM/B$ be the projection, and $x, y \in B$ some vector fields. Then $[fx, y] = f[x, y] - D_y(f)x$. This implies that $\Pi([x, y])$ is $C^\infty(M)$ -linear as a function of x and y .

DEFINITION: The map $[B, B] \rightarrow TM/B$ we have constructed is called **Frobenius bracket** (or **Frobenius form**); it is a skew-symmetric $C^\infty(M)$ -linear form on B with values in TM/B .

DEFINITION: A distribution is called **integrable**, or **holonomic**, or **involutive**, if its Frobenius form vanishes.

Smooth submersions

DEFINITION: Let $\pi : M \longrightarrow M'$ be a smooth map of manifolds. This map is called **submersion** if at each point of M the differential $D\pi$ is surjective, and **immersion** if it is injective.

CLAIM: Let $\pi : M \longrightarrow M'$ be a submersion. Then each $m \in M$ has a neighbourhood $U \cong V \times W$, where V, W are smooth and $\pi|_U$ **is a projection of $V \times W = U \subset M$ to $W \subset M'$ along V .**

EXERCISE: Deduce this result from the inverse function theorem.

EXERCISE: (“Ehresmann’s fibration theorem”)

Let $\pi : M \longrightarrow M'$ be a smooth submersion of compact manifolds. Prove that π is a locally trivial fibration.

DEFINITION: **Vertical tangent space** $T_\pi M \subset TM$ of a submersion $\pi : M \longrightarrow M'$ is the kernel of $D\pi$.

CLAIM: Let $\pi : M \longrightarrow M'$ be a submersion and $T_\pi M \subset TM$ the vertical tangent space. **Then $T_\pi M$ is an involutive subbundle.**

Proof: $D_\pi([X, Y]) = [D_\pi(X), D_\pi(Y)] = 0$ for any $X, Y \in \ker D_\pi$. ■

Frobenius theorem (statement)

Frobenius Theorem: Let $B \subset TM$ be a sub-bundle. Then B is involutive if and only if each point $x \in M$ has a neighbourhood $U \ni x$ and **a smooth submersion $U \xrightarrow{\pi} V$ such that B is its vertical tangent space: $B = T_\pi M$.**

REMARK: The implication “ $B = T_\pi M$ ” \Rightarrow “**Frobenius form vanishes**” was proven above.

DEFINITION: The fibers of π are called **leaves**, or **integral submanifolds** of the distribution B . Globally on M , **a leaf of B** is a maximal connected manifold $Z \hookrightarrow M$ which is immersed to M and tangent to B at each point. A distribution for which Frobenius theorem holds is called **integrable**. If B is integrable, the set of its leaves is called **a foliation**. The leaves are manifolds which are immersed to M , but not necessarily closed.

Frobenius theorem: existence of integral submanifolds

REMARK: To prove the Frobenius theorem for $B \subset TM$, it suffices to show that each point is contained in an integral submanifold. In this case, the smooth submersion $U \xrightarrow{\pi} V$ is a projection to the leaf space of the distribution.

REMARK: When B is 1-dimensional (in this case one says that B has rank 1, denoted $\text{rk } B = 1$), Frobenius theorem follows from existence of the diffeomorphism flow associated with a vector field. Indeed, locally we may assume that B admits a non-degenerate section v . Let $V_t : M \times \mathbb{R} \rightarrow M$ be the corresponding flow of diffeomorphisms. Then $Z_m := V_t(\{m\} \times \mathbb{R})$ is tangent to v everywhere, hence it is a 1-dimensional manifold immersed in M . Clearly, Z_m is a leaf of this distribution. Since B is a tangent to a foliation, it is integrable.

Further on we shall need the following exercise.

EXERCISE: Let $V_t = e^{vt}$ be a diffeomorphism flow on M , and $F \subset TM$ a vector bundle. Assume that $[v_t, F] \subset F$. Prove that then V_t preserves $F \subset TM$.

Basic sub-bundles (1)

DEFINITION: Let $B \subset TM$ be an involutive sub-bundle. A sub-bundle $F \subset TM$ is called **basic** for B if $F \supset B$ and for all $b \in B, b' \in F$, one has $[b, b'] \in F$.

REMARK: One should think of basic sub-bundles as of **sub-bundles preserved by all diffeomorphisms obtained from exponentiation of a vector field $v \in B$.**

LEMMA: Let $B \subset TM$ be an integrable distribution, $\pi : M \rightarrow M_1$ projection to the leaf space of B , and $F \supset B$ a sub-bundle of TM containing B . Then the following conditions are equivalent: **(a) F is basic for B .**
(b) There exists a sub-bundle $F_1 \subset TM_1$ such that $\pi^{-1}(F_1) = F$.

Proof: Next slide.

Basic sub-bundles (2)

LEMMA: Let $B \subset TM$ be an integrable distribution, $\pi : M \rightarrow M_1$ projection to the leaf space of B , and $F \supset B$ a sub-bundle of TM containing B . Then the following conditions are equivalent: **(a) F is basic for B .**

(b) There exists a sub-bundle $F_1 \subset TM_1$ such that $\pi^{-1}F_1 = F$.

Proof. Step 1: Consider coordinates x_1, \dots, x_n on M such that $x_{k+1} = \pi^*(x'_{k+1}, \dots, x_n) = \pi^*(x_n)$, where $x'_i, i = k+1, k+2, \dots, n$ are coordinates on M_1 , and $\frac{d}{dx_1}, \dots, \frac{d}{dx_k}$ generate B . Locally such coordinates always exist, because B is integrable. Denote by G a subgroup of $\text{Diff}(M)$ obtained by exponents of $\frac{d}{dx_1}, \dots, \frac{d}{dx_k}$. Since $[B, F] \subset F$, the corresponding diffeomorphisms preserve F . **Therefore, F is a G -invariant sub-bundle of TM .**

Step 2: Any G -invariant sub-bundle $F \supset B$ is obtained as $\pi^{-1}(F_1)$ for some sub-bundle $F_1 \subset TM_1 = M/G$. Indeed, since the action of G_1 is free, the bundle F is generated over $C^\infty M$ by G -invariant sections. However, any G -invariant bundle F containing B is generated by G -invariant sections, which can be lifted from M/G **(check this)**.

Step 3: Conversely, if F is lifted from $M_1 = M/G$, it is G -invariant, hence $e^{tb}(b') \subset F$, and this gives $[b, b'] \subset F$ **(check this)**. ■

Frobenius theorem (proof)

Frobenius Theorem: Let $B \subset TM$ be a sub-bundle. Then B is involutive if and only if each point $x \in M$ has a neighbourhood $U \ni x$ and **a smooth submersion $U \xrightarrow{\pi} V$ such that B is its vertical tangent space: $B = T_{\pi}M$.**

Proof. Step 1: Consider a rank 1 sub-bundle $B_1 \subset B$. Using the diffeomorphism flow as above, we prove that B_1 is integrable. Since $[B_1, B] \subset B$, the bundle B is basic with respect to B_1 . **Therefore, $B = \pi^{-1}(B')$ for some $B' \subset TM_1$, where M_1 is the leaf space of B_1 .**

Step 2: Let $\pi : M \rightarrow M_1$ be the projection to the leaf space. Then $B = \pi^{-1}(B')$, where $\text{rk } B' = \text{rk } B - 1$. Using induction in $\text{rk } B$, we can assume that B' is integrable. Let $\pi_0 : M_1 \rightarrow M_0$ be the projection to the leaf space of B' , defined locally in M . **Then $\pi \circ \pi_0 : M \rightarrow M_0$ is the projection to the leaf space of B . ■**