Hodge theory

lecture 10: Newlander-Nirenberg theorem

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Almost complex manifolds (reminder)

DEFINITION: Let $I : TM \longrightarrow TM$ be an endomorphism of a tangent bundle satisfying $I^2 = -\operatorname{Id}$. Then I is called **almost complex structure operator**, and the pair (M, I) **an almost complex manifold**.

EXAMPLE: $M = \mathbb{C}^n$, with complex coordinates $z_i = x_i + \sqrt{-1}\, y_i$, and $I(d/dx_i) = d/dy_i$, $I(d/dy_i) = -d/dx_i$.

DEFINITION: Let (V, I) be a space equipped with a complex structure $I : V \longrightarrow V$, $I^2 = -\operatorname{Id}$. **The Hodge decomposition** $V \otimes_{\mathbb{R}} \mathbb{C} := V^{1,0} \oplus V^{0,1}$ is defined in such a way that $V^{1,0}$ is a $\sqrt{-1}$-eigenspace of I, and $V^{0,1}$ a $-\sqrt{-1}$-eigenspace.

DEFINITION: A function $f : M \longrightarrow \mathbb{C}$ on an almost complex manifold is called **holomorphic** if $df \in \Lambda^{1,0}(M)$.

REMARK: For some almost complex manifolds, **there are no holomorphic functions at all**, even locally.

Complex manifolds and almost complex manifolds (reminder)

DEFINITION: Standard almost complex structure is $I(d/dx_i) = d/dy_i$, $I(d/dy_i) = -d/dx_i$ on $\mathbb{C}^n$ with complex coordinates $z_i = x_i + \sqrt{-1}\, y_i$.

DEFINITION: A map $\Psi: (M,I) \longrightarrow (N,J)$ from an almost complex manifold to an almost complex manifold is called **holomorphic** if $\Psi^*(\Lambda^{1,0}(N)) \subset \Lambda^{1,0}(M)$.

REMARK: This is the same as $d\Psi$ being complex linear; for standard almost complex structures, **this is the same as the coordinate components of Ψ being holomorphic functions.**

DEFINITION: A complex manifold is a manifold equipped with an atlas with charts identified with open subsets of $\mathbb{C}^n$ and transition functions holomorphic.

Integrability of almost complex structures (reminder)

DEFINITION: An almost complex structure I on a manifold is called **integrable** if any point of M has a neighbourhood U diffeomorphic to an open subset of $\mathbb{C}^n$, in such a way that the almost complex structure I is induced by the standard one on $U \subset \mathbb{C}^n$.

CLAIM: Complex structure on a manifold M uniquely determines an integrable almost complex structure, and is determined by it.

Proof: Complex structure on a manifold M is determined by the sheaf of holomorphic functions $\mathcal{O}_M$, because $d\mathcal{O}_M$ generates $\Lambda^{1,0}(M)$, and $\mathcal{O}_M$ is determined by I, because $\mathcal{O}_M = \{f \ \mid\ df \in \Lambda^{1,0}(M)\}$. ■

Frobenius form (reminder)

CLAIM: Let $B \subset TM$ be a sub-bundle of a tangent bundle of a smooth manifold. Given vector fiels $X, Y \in B$, consider their commutator $[X,Y]$, and lets $\Psi(X,Y) \in TM/B$ be the projection of $[X,Y]$ to TM/B. **Then $\Psi(X,Y)$ is $C^\infty(M)$-linear in X, Y:**

$$\Psi(fX,Y) = \Psi(X,fY) = f\Psi(X,Y).$$

Proof: Leibnitz identity gives $[X,fY] = f[X,Y] + X(f)Y$, and the second term belongs to B, hence does not influence the projection to TM/B. ■

DEFINITION: This form is called **the Frobenius form** of the sub-bundle $B \subset TM$. This bundle is called **involutive**, or **integrable**, or **holonomic** if $\Psi = 0$.

Formal integrability (reminder)

DEFINITION: An almost complex structure I on (M, I) is called **formally integrable** if $[T^{1,0}M, T^{1,0}] \subset T^{1,0}$, that is, if $T^{1,0}M$ is involutive.

DEFINITION: The Frobenius form $\Psi \in \Lambda^{2,0}M \otimes TM$ is called **the Nijenhuis tensor**.

CLAIM: If a complex structure I on M is integrable, it is formally integrable.

Proof: Locally, the bundle $T^{1,0}(M)$ is generated by d/dz_i, where z_i are complex coordinates. These vector fields commute, hence satisfy $[d/dz_i, d/dz_j] \in T^{1,0}(M)$. This means that the Frobenius form vanishes. ■

THEOREM: (Newlander-Nirenberg)
A complex structure I on M is integrable if and only if it is formally integrable.

Proof: (real analytic case) later today.

REMARK: In dimension 1, formal integrability is automatic. Indeed, $T^{1,0}M$ is 1-dimensional, hence all skew-symmetric 2-forms on $T^{1,0}M$ vanish.

Real analytic manifolds

DEFINITION: **Real analytic function** on an open set $U \subset \mathbb{R}^n$ is a function which admits a Taylor expansion near each point $x \in U$:

$$f(z_1 + t_1, z_2 + t_2, ..., z_n + t_n) = \sum_{i_1,...,i_n} a_{i_1,...,i_n} t_1^{i_1} t_2^{i_2} ... t_n^{i_n}.$$

(here we assume that the real numbers t_i satisfy $|t_i| < \varepsilon$, where ε depends on f and M).

REMARK: Clearly, **real analytic functions constitute a sheaf.**

DEFINITION: A **real analytic manifold** is a ringed space which is locally isomorphic to an open ball $B \subset \mathbb{R}^n$ with the sheaf of real analytic functions.

Involutions

DEFINITION: An involution is a map $\iota: M \longrightarrow M$ such that $\iota^2 = \mathrm{Id}_M$.

EXERCISE: Prove that **any linear involution on a real vector space V is diagonalizable,** with eigenvalues ± 1.

Theorem 1: Let M be a smooth manifold, and $\iota: M \longrightarrow M$ an involutiin. **Then the fixed point set N of ι is a smooth submanifold.**

Proof. Step 1: Inverse function theorem. Let $m \in M$ be a point on a smooth k-dimensional manifold and $f_1, ..., f_k$ functions on M such that their differentials $df_1, ..., df_k$ are linearly independent in m. Then $f_1, ..., f_k$ **define a coordinate system in a neighbourhood of a, giving a diffeomorphism of this neighbourhood to an open ball.**

Step 2: Assume that $d\iota$ has k eigenvalues 1 on $T_m M$, and $n-k$ eigenvalues -1. Choose a coordinate system $x_1, ..., x_n$ on M around a point $m \in N$ such that $dx_1|_m, ..., dx_k|_m$ are ι-invariant and $dx_{k+1}|_m, ..., dx_n|_m$ are ι-anti-invariant. Let $y_1 = x_1 + \iota^* x_1$, $y_2 = x_2 + \iota^* x_2$, ... $y_k = x_k + \iota^* x_k$, and $y_{k+1} = x_{k+1} - \iota^* x_{k+1}$, $y_{k+2} = x_{k+2} - \iota^* x_{k+2}$, ... $y_n = x_n - \iota^* x_n$. Since $dx_i|_m = xy_i|_m$, these differentials are linearly independent in m. By Step 1, **functions y_i define an ι-invariant coordinate system on an open neighbourhood of m, with N given by equations $y_{k+1} = y_{k+2} = ... = y_n = 0$.** ■

Real structures

DEFINITION: An involution is a map $\iota : M \longrightarrow M$ such that $\iota^2 = \mathrm{Id}_M$. **A real structure** on a complex vector space $V = \mathbb{C}^n$ is an $\mathbb{R}$-linear involution $\iota : V \longrightarrow V$ such that $\iota(\lambda x) = \overline{\lambda}\iota(x)$ for any $\lambda \in \mathbb{C}$.

DEFINITION: A map $\Psi : M \longrightarrow M$ on an almost complex manifold (M, I) is called **antiholomorphic** if $d\Psi(I) = -I$. A function f is called **antiholomorphic** if $\overline{f}$ is holomorphic.

EXERCISE: Prove that **antiholomorphic function on M defines an antiholomorphic map from M to $\mathbb{C}$.**

EXERCISE: Let ι be a smooth map from a complex manifold M to itself. Prove that **ι is antiholomorphic if and only if $\iota^*(f)$ is antiholomorphic for any holomorphic function f on $U \subset M$.**

DEFINITION: A real structure on a complex manifold M is an antiholomorphic involution $\tau : M \longrightarrow M$.

EXAMPLE: Complex conjugation defines a real structure on $\mathbb{C}^n$.

Real analytic manifolds and real structures

PROPOSITION: Let $M_{\mathbb R}\subset M_{\mathbb C}$ be a fixed point set of an antiholomorphic involution ι, U_i a complex analytic atlas, and $\Psi_{ij}: U_{ij}\longrightarrow U_{ij}$ the gluing functions. **Then, for an appropriate choice of coordinate systems all Ψ_{ij} are real analytic on $M_{\mathbb R}$, and define a real analytic atlas on the manifold $M_{\mathbb R}$.**

Proof. Step 1: Let $z_1, ..., z_n$ be a holomorphic coordinate system on $M_{\mathbb C}$ in a neighbourhood of $m\in M_{\mathbb R}$ such that $\iota(dz_i)=d\overline{z}_i$ in $T^*_m M$. Such a coordinate system can be chosen by taking linear functions with prescribed differentials in m. **Replacing z_i by $x_i := z_i+\iota^*(\overline{z}_i)$, we obtain another coordinate system x_i on M** (compare with Theorem 1).

Step 2: This new coordinate system satisfies $\iota^* x_i=\overline{x}_i$, hence $M_{\mathbb R}$ in these coordinates is giving by equation $\operatorname{im} x_1=\operatorname{im} x_2=...=\operatorname{im} x_n=0$. **All gluing functions from such coordinate system to another one of this type satisfy $\Psi_{ij}(\overline{z}_i)=\overline{\Psi_{ij}(\overline{z}_i)}$, hence they are real on $M_{\mathbb R}$.** ■

Real analytic manifolds and real structures (2)

PROPOSITION: Any real analytic manifold can be obtained from this construction.

Proof. Step 1: Let $\{U_i\}$ be a locally finite atlas of a real analytic manifold M, and $\Psi_{ij}: U_{ij} \longrightarrow U_{ij}$ the gluing map. We realize U_i as an open ball with compact closure in $\operatorname{Re}(\mathbb{C}^n) = \mathbb{R}^n$. By local finiteness, there are only finitely many such Ψ_{ij} for any given U_i. Denote by B_ε an open ball of radius ε in the n-dimensional real space $\operatorname{im}(\mathbb{C}^n)$.

Step 2: Let $\varepsilon > 0$ be a sufficiently small real number such that all Ψ_{ij} can be extended to gluing functions $\tilde{\Psi}_{ij}$ on the open sets $\tilde{U}_i := U_i \times B_\varepsilon \subset \mathbb{C}^n$. **Then $(\tilde{U}_i, \Psi_{ij})$ is an atlas for a complex manifold $M_{\mathbb{C}}$.** Since all Ψ_{ij} are real, they are preserved by natural involution acting on B_ε as -1 and on U_i as identity. This involution defines a real structure on $M_{\mathbb{C}}$. Clearly, M is the set of its fixed points. ■

Complexification

DEFINITION: Let $M_{\mathbb{R}}$ be a real analytic manifold, and $M_{\mathbb{C}}$ a complex analytic manifold equipped with an antiholomorphic involution, such that $M_{\mathbb{R}}$ is the set of its fixed points. Then $M_{\mathbb{C}}$ is called **complexification** of $M_{\mathbb{R}}$.

DEFINITION: A tensor on a real analytic manifold is called **real analytic** if it is expressed locally by a sum of coordinate monomials with real analytic coefficients.

CLAIM: Let $M_{\mathbb{R}}$ be a real analytic manifold, $M_{\mathbb{C}}$ its complexification, and Φ a tensor on $M_{\mathbb{R}}$. **Then Φ is real analytic if and only if Φ can be extended to a holomorpic tensor $\Phi_{\mathbb{C}}$ in some neighbourhood of $M_{\mathbb{R}}$ inside $M_{\mathbb{C}}$.**

Proof: The "if" part is clear, because every complex analytic tensor on $M_{\mathbb{C}}$ is by definition real analytic on $M_{\mathbb{R}}$.

Conversely, suppose that Φ is expressed by a sum of coordinate monomials with real analytic coefficients f_i. Let $\{U_i\}$ be a cover of M, and $\tilde{U}_i := U_i \times B_\varepsilon$ the corresponding cover of a neighbourhood of $M_{\mathbb{R}}$ in $M_{\mathbb{C}}$ constructed above. Chosing ε sufficiently small, we can assume that the Taylor series giving coefficients of Φ converges on each $\tilde{U}_i$. **We define $\Phi_{\mathbb{C}}$ as the sum of these series.** ■

Extension of tensors to a complexification

Lemma 1: Let X be an open ball in $\mathbb{C}^n$ equipped with the standard anticomplex involution, $X_{\mathbb{R}} = X \cap \mathbb{R}^n$ its fixed point set, and α a holomorphic tensor on X vanishing in $X_{\mathbb{R}}$. **Then $\alpha = 0$.**

Proof: Any holomorphic function which vanishes on $\mathbb{R}^n$ has all its derivatives is equal zero. Therefore its Taylor series vanish. Such a function vanishes on $\mathbb{C}^n$ by analytic continuation principle. This argument can be applied to all coefficients of α. ■

DEFINITION: An almost complex structure I on a real analytic manifold is **real analytic** if I is a real anaytic tensor.

COROLLARY: Let (M, I) be a real analytic almost complex manifold, $M_{\mathbb{C}}$ its complexification, and $I_{\mathbb{C}} : TM_{\mathbb{C}} \longrightarrow TM_{\mathbb{C}}$ the holomorphic extension of I to $M_{\mathbb{C}}$. **Then $I_{\mathbb{C}}^2 = -\operatorname{Id}$.**

Proof: The tensor $I_{\mathbb{C}}^2 + \operatorname{Id}$ is holomorphic and vanishes on $M_{\mathbb{R}}$, hence the previous lemma can be applied. ■

Underlying real analytic manifold

REMARK: A complex analytic map $\Phi: \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is real analytic as a map $\mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$. Indeed, the coefficients of Φ are real and imaginary parts of holomorphic functions, and real and imaginary parts of holomorphic functions can be expressed as Taylor series of the real variables.

DEFINITION: Let M be a complex manifold. The **underlying real analytic manifold** is the same manifold, with the same gluing functions, considered as real analytic maps.

DEFINITION: Let M be a complex manifold. The **complex conjugate manifold** is the same manifold with almost complex structure $-I$ and anti-holomorphic functions on M holomorphic on $\overline{M}$.

CLAIM: Let M be an integrable almost complex manifold. Denote by $M_{\mathbb{R}}$ its underlying real analytic manifold. **Then a complexification of $M_{\mathbb{R}}$ can be given as $M_{\mathbb{C}} := M \times \overline{M}$,** with the anticomplex involution $\tau(x, y) = (y, x)$.

Proof: Clearly, the fixed point set of τ is the diagonal, identified with $M_{\mathbb{R}} = M$ as usual. Both holomorphic and antiholomorphic functions on $M_{\mathbb{R}}$ are obtained as restrictions of holomorphic functions from $M_{\mathbb{C}}$, hence the sheaf of real analytic functions on $M_{\mathbb{R}}$ is a real part of the sheaf $\mathcal{O}_{M_{\mathbb{C}}}$ of holomorphic functions on $M_{\mathbb{C}}$. ■

Holomorphic and antiholomorphic foliations

DEFINITION: Let $B\subset TM$ be a sub-bundle. The **foliation associated with B** is a family of submanifolds $X_t\subset U$, defined for each sufficiently small subset of M, called **the leaves of the foliation**, such that B is the bundle of vectors tangent to X_t. In this case, X_t are called **the leaves** of the foliation.

Frobenius Theorem: **B is involutive if and only if it is tangent to a foliation.**

REMARK: Let (M,I) be a real analytic almost complex manifold, and $M_{\mathbb C}$ its complexification. Replacing $M_{\mathbb C}$ by a smaller neighbourhood of M, we may assume that the tensor I is extended to an endomorphism $I: TM_{\mathbb C} \longrightarrow TM_{\mathbb C}$, $I^2=-\operatorname{Id}$. **Since $TM_{\mathbb C}$ is a complex vector bundle, I acts there with the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, giving a decomposition $TM_{\mathbb C}=T^{1,0}M_{\mathbb C}\oplus T^{0,1}M_{\mathbb C}$**

DEFINITION: **Holomorphic foliation** is a foliation tangent to $T^{1,0}M_{\mathbb C}$, **antiholomorphic foliation** is a foliation tangent to $T^{0,1}M_{\mathbb C}$.

Antiholomorphic foliation on $M_{\mathbb C} = M \times \overline{M}$.

CLAIM: Let (M, I) be a integrable almost complex manifold, $M_{\mathbb C} = M \times \overline{M}$ its complexification, and $\pi, \overline{\pi}$ projections of $M_{\mathbb C}$ to M and $\overline{M}$. **Then the fibers of $\overline{\pi}$ is a holomorphic foliation, and the fibers of π is a holomorphic foliation.**

Proof: Let $TM_{\mathbb C} = T' \oplus T''$ be a decomposition of $TM_{\mathbb C}$ onto part tangent to fibers of $\overline{\pi}$ and tangent to fibers of π. **On $M_{\mathbb R}$ the decomposition $TM_{\mathbb C} = T' \oplus T''$ coincides with the decomposition $TM \otimes \mathbb C = T^{1,0}M \oplus T^{0,1}M$.** By Lemma 1 the same is true everywhere on $M_{\mathbb C}$. ■

COROLLARY: Let (M, I) be a integrable almost complex manifold. **Then I is a real analytic almost complex structure.**

Proof: It was extended to $M_{\mathbb C}$ in the previous claim. ■

Corollary 1: Let (M, I) be a real analytic almost complex manifold. Then holomorphic functions on $M_{\mathbb C}$ which are constant on the leaves of antiholomorphic foliation **restrict to holomorphic functions on $(M, I) \subset M_{\mathbb C}$.**

Proof: Such functions are constant in the $(0,1)$-direction on $TM \otimes \mathbb C$. ■

Integrability of real analytic almost complex structures

THEOREM: Let (M,I) be a real analytic almost complex manifold. **Then M is integrable.**

Proof. Step 1: Consider the complexification $M_{\mathbb{C}}$ of M, and let $TM_{\mathbb{C}} = T^{1,0}M_{\mathbb{C}} \oplus T^{0,1}M_{\mathbb{C}}$ be the decomposition defined above. By Frobenius theorem, there exists a foliation tangent to $T^{0,1}M_{\mathbb{C}}$ and one tangent to $T^{1,0}M_{\mathbb{C}}$. Since the leaves of these foliations are transversal, **locally $M_{\mathbb{C}}$ is a product of M' and M'' which are identified with the space of leaves of $T^{0,1}M_{\mathbb{C}}$ and $T^{1,0}M_{\mathbb{C}}$.**

Step 2: Locally, functions on M' can be lifted to $M' \times M'' = M_{\mathbb{C}}$, giving functions which are constant on the leaves of the foliation tangent to $T^{0,1}M_{\mathbb{C}}$. By Corollary 1, such functions are holomorphic on (M,I). Choosing a function with linearly independent differentials in $x \in M$, it would give a **holomorphic coordinate system in a neigbourhood of (M,I),** and the transition functions between such coordinate systems are by construction holomorphic.

■