Hodge theory

lecture 11: Kähler manifolds

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Almost complex manifolds (reminder)

DEFINITION: Let $I: TM \longrightarrow TM$ be an endomorphism of a tangent bundle satisfying $I^2 = -\operatorname{Id}$. Then I is called **almost complex structure operator**, and the pair (M, I) **an almost complex manifold**.

EXAMPLE: $M = \mathbb{C}^n$, with complex coordinates $z_i = x_i + \sqrt{-1}\, y_i$, and $I(d/dx_i) = d/dy_i$, $I(d/dy_i) = -d/dx_i$.

DEFINITION: Let (V, I) be a space equipped with a complex structure $I: V \longrightarrow V$, $I^2 = -\operatorname{Id}$. **The Hodge decomposition** $V \otimes_{\mathbb{R}} \mathbb{C} := V^{1,0} \oplus V^{0,1}$ is defined in such a way that $V^{1,0}$ is a $\sqrt{-1}$-eigenspace of I, and $V^{0,1}$ a $-\sqrt{-1}$-eigenspace.

DEFINITION: A function $f: M \longrightarrow \mathbb{C}$ on an almost complex manifold is called **holomorphic** if $df \in \Lambda^{1,0}(M)$.

REMARK: For some almost complex manifolds, **there are no holomorphic functions at all**, even locally.

Complex manifolds and almost complex manifolds (reminder)

DEFINITION: **Standard almost complex structure** is $I(d/dx_i) = d/dy_i$, $I(d/dy_i) = -d/dx_i$ on $\mathbb{C}^n$ with complex coordinates $z_i = x_i + \sqrt{-1}\, y_i$.

DEFINITION: A map $\Psi: (M,I) \longrightarrow (N,J)$ from an almost complex manifold to an almost complex manifold is called **holomorphic** if $\Psi^*(\Lambda^{1,0}(N)) \subset \Lambda^{1,0}(M)$.

REMARK: This is the same as $d\Psi$ being complex linear; for standard almost complex structures, **this is the same as the coordinate components of Ψ being holomorphic functions.**

DEFINITION: **A complex manifold** is a manifold equipped with an atlas with charts identified with open subsets of $\mathbb{C}^n$ and transition functions holomorphic.

Integrability of almost complex structures (reminder)

DEFINITION: An almost complex structure I on a manifold is called **integrable** if any point of M has a neighbourhood U diffeomorphic to an open subset of $\mathbb{C}^n$, in such a way that the almost complex structure I is induced by the standard one on $U \subset \mathbb{C}^n$.

CLAIM: Complex structure on a manifold M uniquely determines an integrable almost complex structure, and is determined by it.

Proof: Complex structure on a manifold M is determined by the sheaf of holomorphic functions $\mathcal{O}_M$, because $d\mathcal{O}_M$ generates $\Lambda^{1,0}(M)$, and $\mathcal{O}_M$ is determined by I, because $\mathcal{O}_M = \{f \mid df \in \Lambda^{1,0}(M)\}$. ∎

Formal integrability (reminder)

DEFINITION: An almost complex structure I on (M, I) is called **formally integrable** if $[T^{1,0}M, T^{1,0}] \subset T^{1,0}$, that is, if $T^{1,0}M$ is involutive.

DEFINITION: The Frobenius form $\Psi \in \Lambda^{2,0}M \otimes TM$ is called **the Nijenhuis tensor**.

CLAIM: If a complex structure I on M is integrable, it is formally integrable.

Proof: Locally, the bundle $T^{1,0}(M)$ is generated by d/dz_i, where z_i are complex coordinates. These vector fields commute, hence satisfy $[d/dz_i, d/dz_j] \in T^{1,0}(M)$. This means that the Frobenius form vanishes. ■

THEOREM: (Newlander-Nirenberg)
A complex structure I on M is integrable if and only if it is formally integrable.

REMARK: In dimension 1, formal integrability is automatic. Indeed, $T^{1,0}M$ is 1-dimensional, hence all skew-symmetric 2-forms on $T^{1,0}M$ vanish.

Riemannian manifolds

DEFINITION: Let $h \in \operatorname{Sym}^2 T^*M$ be a symmetric 2-form on a manifold which satisfies $h(x,x) > 0$ for any non-zero tangent vector x. Then h is called **Riemannian metric**, of **Riemannian structure**, and (M,h) **Riemannian manifold**.

DEFINITION: For any $x, y \in M$, and any path $\gamma: [a,b] \longrightarrow M$ connecting x and y, consider **the length** of γ defined as $L(\gamma) = \int_\gamma |\frac{d\gamma}{dt}| dt$, where $|\frac{d\gamma}{dt}| = h(\frac{d\gamma}{dt}, \frac{d\gamma}{dt})^{1/2}$. Define **the geodesic distance** as $d(x,y) = \inf_\gamma L(\gamma)$, where infimum is taken for all paths connecting x and y.

EXERCISE: Prove that the **geodesic distance satisfies triangle inequality and defines metric on M.**

EXERCISE: Prove that **this metric induces the standard topology on M.**

EXAMPLE: Let $M = \mathbb{R}^n$, $h = \sum_i dx_i^2$. **Prove that the geodesic distance coincides with $d(x,y) = |x-y|$.**

EXERCISE: Using partition of unity, **prove that any manifold admits a Riemannian structure.**

Kähler manifolds

DEFINITION: An Riemannian metric g on an almost complex manifiold M is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

REMARK: Given any Riemannian metric g on an almost complex manifold, **a Hermitian metric h can be obtained as $h = g + I(g)$, where $I(g)(x, y) = g(I(x), I(y))$.**

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

REMARK: It is $U(1)$-invariant, hence **of Hodge type (1,1)**.

REMARK: In the triple I, g, ω, **each element can recovered from the other two.**

DEFINITION: A complex Hermitian manifold (M, I, ω) is called **Kähler** if $d\omega = 0$. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M, and ω **the Kähler form**.

Homogeneous spaces

DEFINITION: A Lie group is a smooth manifold equipped with a group structure such that the group operations are smooth. Lie group G **acts on a manifold** M if the group action is given by the smooth map $G \times M \longrightarrow M$.

DEFINITION: Let G be a Lie group acting on a manifold M transitively. Then M is called **a homogeneous space**. For any $x \in M$ the subgroup $\operatorname{St}_x(G) = \{g \in G \mid g(x) = x\}$ is called **stabilizer of a point** x, or **isotropy subgroup**.

CLAIM: For any homogeneous manifold M with transitive action of G, **one has $M = G/H$,** where $H = \operatorname{St}_x(G)$ is an isotropy subgroup.

Proof: The natural surjective map $G \longrightarrow M$ putting g to $g(x)$ identifies M with the space of conjugacy classes G/H. ■

REMARK: Let $g(x) = y$. Then $\operatorname{St}_x(G)^g = \operatorname{St}_y(G)$: **all the isotropy groups are conjugate.**

Isotropy representation

DEFINITION: Let $M = G/H$ be a homogeneous space, $x \in M$ and $\mathrm{St}_x(G)$ the corresponding stabilizer group. The **isotropy representation** is the natural action of $\mathrm{St}_x(G)$ on T_xM.

DEFINITION: A tensor Φ on a homogeneous manifold $M = G/H$ is called **invariant** if it is mapped to itself by all diffeomorphisms which come from $g \in G$.

REMARK: Let Φ_x be an isotropy invariant tensor on $\mathrm{St}_x(G)$. For any $y \in M$ obtained as $y = g(x)$, consider the tensor Φ_y on T_yM obtained as $\Phi_y := g(\Phi)$. The choice of g is not unique, however, for another $g' \in G$ which satisfies $g'(x) = y$, we have $g = g'h$ where $h \in \mathrm{St}_x(G)$. Since Φ is h-invariant, **the tensor Φ_y is independent from the choice of g.**

We proved

Theorem 1: G-invariant tensors on $M = G/H$ are in bijective correspondence with isotropy invariant tensors on T_xM, for any $x \in M$. ■

Representations acting transitively on a sphere

THEOREM: Let G be a group acting on a vector space V. Suppose that G acts transitively on a unit sphere $\{x \in V \ \mid \ g(x) = 1\}$. **Then a G-invariant bilinear symmetric form is unique up to a constant multiplier.**

Proof. Step 1: Since G preserves the sphere, which is a level set of the quadratic form g, g is G-invariant.

Step 2: For any G-invariant quadratic form g', the function $x \longrightarrow \frac{g'(x)}{g(x)}$ is constant on spheres and invariant under homothety, hence it is constant. ■

EXERCISE: Let V be a representation of G, and suppose G acts transitively on a sphere. **Prove that V is an irreducible representation.**

EXERCISE: Prove the **Schur lemma:** let V be an irreducible representation of G over $\mathbb{R}$, and g a G-invariant positive definite bilinear symmetric form. **Then any G-invariant bilinear symmetric form is proportional to g.**

Fubini-Study form

EXAMPLE: Consider the natural action of the unitary group $U(n+1)$ on $\mathbb{C}P^n$. The stabilizer of a point is $U(n) \times U(1)$.

THEOREM: There exists an $U(n+1)$-invariant Riemann form on $\mathbb{C}P^n$. Moreover, **such a form is unique up to a constant multiplier, and Kähler.**

REMARK: This Riemannian structure is called **the Fubini-Study metric**, and its Hermitian form **the Fubini-Study form**.

Proof. Step 1: To construct a $U(n+1)$-invariant Riemann form on $\mathbb{C}P^n$, we take a $U(n)$-invariant form on $T_x\mathbb{C}P^n$ and apply Theorem 1. A $U(n)$-invariant form on $T_x\mathbb{C}P^n$ exists, because it is a standard representation.

Step 2: Uniqueness follows because the isotropy group acts transitively on a sphere. ■

CLAIM: The Fubini-Study form is closed, and the corresponding metric is Kähler.

Proof: Let ω be a Fubini-Study form. Then $d\omega$ is an isotropy-invariant 3-form on $T_x\mathbb{C}P^n$. However, the isotropy group contains $-\operatorname{Id}$, **hence all isotropy-invariant odd tensors vanish.** ■

Projective manifolds

DEFINITION: Let M be a complex manifold, and $X \subset M$ a smooth submanifold. It is called **a complex submanifold** if $I(TX) \subset TX$, and the map $X \hookrightarrow M$ **a complex embedding**. A complex manifold which admits a complex embedding to $\mathbb{C}P^n$ is called **a projective manifold**.

REMARK: A complex submanifold of a Kähler manifold is Kähler. Indeed, restriction of a Hermitian metric is Hermitian, and restriction of a closed form is closed. Therefore, **all projective manifolds are Kähler**.

DEFINITION: A subvariety of $\mathbb{C}P^n$ is called **complex algebraic** if can be obtained as common zeroes of a system of homogeneous polynomial equations.

THEOREM: (Chow theorem) All complex submanifolds in $\mathbb{C}P^n$ are complex algebraic.

Kodaira embedding theorem

DEFINITION: Kähler class of a Kähler manifold is the cohomology class $[\omega] \in H^2(M,\mathbb{R})$ of its Kähler form. We say that M **has integer Kähler class** if $[\omega]$ belongs to the image of $H^2(M,\mathbb{Z})$ in $H^2(M,\mathbb{R})$

REMARK: $H^2(\mathbb{C}P^n,\mathbb{R}) = \mathbb{R}$. This implies that **the cohomology class of Fubini-Study form can be chosen integer.** In particular, **all projective manifolds admit Kähler structures with integer Kähler classes.**

THEOREM: (Kodaira embedding theorem) Let M be a Kähler manifold with an integer Kähler class. **Then it is projective.**

This theorem will be proven later in these lectures.

Classes of almost complex manifolds