# Hodge theory

## lecture 12: Levi-Civita connection

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## Riemannian manifolds (reminder)

**DEFINITION:** Let $h \in \operatorname{Sym}^2 T^*M$ be a symmetric 2-form on a manifold which satisfies $h(x,x) > 0$ for any non-zero tangent vector $x$. Then $h$ is called **Riemannian metric**, of **Riemannian structure**, and $(M,h)$ **Riemannian manifold**.

**DEFINITION:** For any $x, y \in M$, and any path $\gamma: [a,b] \longrightarrow M$ connecting $x$ and $y$, consider **the length** of $\gamma$ defined as $L(\gamma) = \int_\gamma |\frac{d\gamma}{dt}| dt$, where $|\frac{d\gamma}{dt}| = h(\frac{d\gamma}{dt}, \frac{d\gamma}{dt})^{1/2}$. Define **the geodesic distance** as $d(x,y) = \inf_\gamma L(\gamma)$, where infimum is taken for all paths connecting $x$ and $y$.

**EXAMPLE:** Let $M = \mathbb{R}^n$, $h = \sum_i dx_i^2$. **Prove that the geodesic distance coincides with $d(x,y) = |x-y|$.**

**EXERCISE:** Using partition of unity, **prove that any manifold admits a Riemannian structure.**

## Connections (reminder)

**DEFINITION:** Recall that **a connection** on a bundle $B$ is an operator $\nabla: B \longrightarrow B \otimes \Lambda^1 M$ satisfying $\nabla(fb) = b \otimes df + f\nabla(b)$, where $f \longrightarrow df$ is de Rham differential. When $X$ is a vector field, we denote by $\nabla_X(b) \in B$ the term $\langle \nabla(b), X \rangle$.

**REMARK: A connection $\nabla$ on $B$ gives a connection $B^* \xrightarrow{\nabla^*} \Lambda^1 M \otimes B^*$ on the dual bundle,** by the formula

$$d(\langle b, \beta \rangle) = \langle \nabla b, \beta \rangle + \langle b, \nabla^* \beta \rangle$$

These connections are usually denoted **by the same letter $\nabla$.**

**REMARK:** For any tensor bundle $\mathcal{B}_1 := B^* \otimes B^* \otimes ... \otimes B^* \otimes B \otimes B \otimes ... \otimes B$ **a connection on $B$ defines a connection on $\mathcal{B}_1$** using the Leibniz formula:

$$\nabla(b_1 \otimes b_2) = \nabla(b_1) \otimes b_2 + b_1 \otimes \nabla(b_2).$$

## Parallel transport along the connection

**THEOREM:** Let $B$ be a vector bundle with connection over $\mathbb{R}$. Then for each $x \in \mathbb{R}$ and each vector $b_x \in B|_x$ **there exists a unique section $b \in B$ such that $\nabla b = 0$, $b|_x = b_x$.**

**Proof:** This is existence and uniqueness of solutions of an ODE $\frac{db}{dt} + A(b) = 0$. ■

**DEFINITION:** Let $\gamma: [0,1] \longrightarrow M$ be a smooth path in $M$ connecting $x$ and $y$, and $(B, \nabla)$ a vector bundle with connection. Restricting $(B, \nabla)$ to $\gamma([0,1])$, we obtain a bundle with connection on an interval. Solve an equation $\nabla(b) = 0$ for $b \in B\Big|_{\gamma([0,1])}$ and initial condition $b|_x = b_x$. This process is called **parallel transport** along the path via the connection. The vector $b_y := b|_y$ is called **vector obtained by parallel transport of $b_x$ along $\gamma$**. **Holonomy group** of $\gamma$ is the group of endomorphisms of the fiber $B_x$ obtained from parallel transports along all paths starting and ending in $x \in M$

## Parallel tensors

**DEFINITION:** Let $B$ be a vector bundle, and $\Psi \in B^{\otimes i} \otimes (B^*)^{\otimes j}$ a tensor on $B$. We say that **connection $\nabla$ preserves $\Psi$** if $\nabla(\Psi) = 0$. In this case we also say that $\Psi$ is **parallel** with respect to the connection.

**REMARK:** $\nabla(\Psi) = 0$ is equivalent to $\Psi$ being a solution of $\nabla(\Psi) = 0$ on each path $\gamma$. This means that **parallel transport preserves $\Psi$.**

We obtained

**COROLLARY: A tensor is parallel if and only if it is holonomy invariant.**

**EXAMPLE: Orthogonal connection**: given a positive definite form $h \in \operatorname{Sym}^2 B^*$ on $B$, a connection $\nabla$ such that $\nabla(h) = 0$ is called **orthogonal**.

**EXAMPLE:** Suppose that $(B, I)$ is a complex vector bundle equipped with a Hermitian metric $h$. A connection $\nabla$ such that $\nabla(I) = \nabla(h) = 0$ is called **unitary**, or **Hermitian**.

## Torsors and affine spaces

**DEFINITION: A torsor** over a group $G$ is a space $X$ equipped with a free and transitive action of $G$, $g, x \longrightarrow \rho(g,x)$.

**DEFINITION: Morphism** of torsors $(X, G, \rho) \xrightarrow{\Psi} (X', G', \rho')$ is a pair $\Psi_X : X \longrightarrow X', \Psi_G : \quad G \longrightarrow G'$, where $\Psi_G$ is a group homomorphism satisfying $\Psi_X(\rho(g,x)) = \rho'(\Psi_G(g), \Psi_X(x))$ (that is, compatible with the map $\Psi_X$).

**REMARK: This defines the category of torsors.**

**DEFINITION: Affine space** is a torsor over a vector space $V$, which is called **linearization**. The action of $V$ on $A$ is denoted $a, v \longrightarrow a + v$. **Morphism** of affine spaces is the morphism of the corresponding torsors.

**REMARK:** Morphism of affine spaces a map $A \xrightarrow{\Psi_A} A'$ and a homomorphism of their linearizations $V \xrightarrow{\Psi_V} V'$ such that $\Psi_A(a + l) = \Psi_A(a) + \Psi_L(l)$.

**EXAMPLE:** Given two connections $\nabla$ and $\nabla_1$ on $B$, the difference $\nabla - \nabla_1$ is an End$(B)$-valued 1-form. Converse is also true: for any End$(B)$-valued 1-form $A \in \Lambda^1 M \otimes \operatorname{End}(B)$, the operator $\nabla + A$ is a connection. In other words, **the space of connections is an affine space over $\Lambda^1 M \otimes \operatorname{End}(B)$**.

## Lie algebra and tensors

**DEFINITION:** Let $V$ be a representation of a Lie algebra $\mathfrak{g}$. **Then $V^*$ is also a representation;** the action of $\mathfrak{g}$ on $V^*$ is given by the formula $\langle g(x), \lambda\rangle = -\langle x, g(\lambda)\rangle$, for all $x \in V, \lambda \in V^*$. A tensor product of two $\mathfrak{g}$-representations $V_1, V_2$ is also a $\mathfrak{g}$-representation, with the action of $\mathfrak{g}$ defined by $g(x \otimes y) = g(x) \otimes y + x \otimes g(y)$. This defines the action of $\mathfrak{g}$ on all tensor powers $V^{\otimes i} \otimes (V^*)^{\otimes j}$, which are called **the tensor representations** of $\mathfrak{g}$. We say that $\mathfrak{g}$ **preserves a tensor $\Phi$** if $g(\Psi) = 0$ for all $g \in \mathfrak{g}$.

**EXAMPLE:** The algebra of all $g \in \operatorname{End}(V)$ preserving a non-degenerate bilinear symmetric form $h \in \operatorname{Sym}^2(V^*)$ is called **orthogonal algebra**, denoted $\mathfrak{so}(V, h)$ or $\mathfrak{so}(V)$. Since $g \in \mathfrak{so}(V)$ if and only if $h(g(x), y) = -h(x, g(y))$, **$\mathfrak{so}(V)$ is represented by antisymmetric matrices.**

**CLAIM:** Let $h \in \operatorname{Sym}^2(V^*)$ be a non-degenerate bilinear symmetric form. Using $h$, we identify $V$ and $V^*$. This gives an isomorphism $V^* \otimes V^* \xrightarrow{\tau} V^* \otimes V = \operatorname{End}(V)$. **Then $\tau(\Lambda^2 V^*) = \mathfrak{so}(V)$.**

**Proof:** For any $f \in \operatorname{End}(V)$, the 2-form $\tau^{-1}(f)$ is written as $x, y \longrightarrow h(f(x), y)$. By definition, $f \in \mathfrak{so}(V)$ means that $h(f(x), y) = -h(x, f(y))$ and this happens if and only if $\tau^{-1}(f)$ is antisymmetric. ∎

## The Lie algebra $\mathfrak{u}(V)$

**EXAMPLE:** Let $(V, I)$ be a real vector space with a complex structure map $I: V \longrightarrow V$, $I^2 = -\operatorname{Id}$, and a Hermitian (that is, $I$-invariant) scalar product. Define **the unitary Lie algebra** $\mathfrak{u}(V) = \{f \in \operatorname{End}(V) \ \mid \ f(I) = f(h) = 0\}$. This is the same as the space of $I$-invariant orthogonal matrices.

**CLAIM:** Consider the natural map $V^* \otimes V^* \xrightarrow{\tau} V^* \otimes V = \operatorname{End}(V)$ associated with $h$. **Then $\tau(\Lambda^{1,1}(V^*)) = \mathfrak{u}(V)$.**

**Proof:** The isomorphism $\tau$ is $I$-invariant, because $h$ is $I$-invariant. **Then $\tau^{-1}(\mathfrak{u}(V))$ is the space of $I$-invariant 2-forms,** which is precisely $\Lambda^{1,1}(V^*)$.
■

## Affine space of orthogonal connections

**CLAIM:** Let $B$ be a bundle with a scalar product. Then **the space of orthogonal connections on $B$ an affine space over $\Lambda^1 M \otimes \mathfrak{so}(B)$.**

**Proof:** Let $s \in B^* \otimes B^*$ be a 2-form on $B$. The action of $A := \nabla - \nabla_1$ on $B^* \otimes B^*$ is given by $A(s)(x,y) = -s(A(x),y) - s(x, A(y))$. Therefore, a difference $A$ of orthogonal connections satisfies $h(A(x),y) = -h(x, A(y))$ for all $x, y \in B$. This is the same as $A \in \Lambda^1 M \otimes \mathfrak{so}(B)$. ■

Similarly one proves

**CLAIM:** Let $B$ be a bundle with a Hermitian structure product. Then **the space of orthogonal connections on $B$ an affine space over $\Lambda^1 M \otimes \mathfrak{u}(B)$.**

**CLAIM:** Let $B$ be a bundle with a Hermitian structure and a tensor $\Phi$, and $\mathfrak{g} \subset \operatorname{End}(B)$ the Lie algebra of endomorphisms preserving $\Phi$. Then **the space of connections on $B$ preserving $\Phi$ is an affine space over $\Lambda^1 M \otimes \mathfrak{g}$.**

## REMINDER: de Rham algebra

**DEFINITION:** Let $\Lambda^* M$ denote the vector bundle with the fiber $\Lambda^* T_x^* M$ at $x \in M$ ($\Lambda^* T^* M$ is the Grassman algebra of the cotangent space $T_x^* M$). The sections of $\Lambda^i M$ are called **differential $i$-forms**. The algebraic operation "wedge product" defined on differential forms is $C^\infty M$-linear; the space $\Lambda^* M$ of all differential forms is called **the de Rham algebra**.

**REMARK:** $\Lambda^0 M = C^\infty M$.

**THEOREM:** **There exists a unique operator $C^\infty M \xrightarrow{d} \Lambda^1 M \xrightarrow{d} \Lambda^2 M \xrightarrow{d} \Lambda^3 M \xrightarrow{d} \ldots$ satisfying the following properties**

1. On functions, $d$ is equal to the differential.
2. $d^2 = 0$
3. $d(\eta \wedge \xi) = d(\eta) \wedge \xi + (-1)^{\tilde\eta} \eta \wedge d(\xi)$, where $\tilde\eta = 0$ where $\eta \in \lambda^{2i} M$ is **an even form,** and $\eta \in \lambda^{2i+1} M$ is **odd.**

**DEFINITION:** The operator $d$ is called **de Rham differential**.

## Cartan formula

**CLAIM:** For any $\eta \in \Lambda^1 M$, and $X, Y \in TM$ one has

$$d\eta(X,Y) = \eta([X,Y]) - \operatorname{Lie}_X(\eta(Y)) + \operatorname{Lie}_Y(\eta(X)).$$

**Proof:** Two sides of this equation define two operators $d, d_1 \Lambda^1 M \longrightarrow \Lambda^2 M$. Both operators satisfy the Leibniz rule $d(f\eta) = df \wedge d\eta + f d\eta$. When $\eta = df$ is exact, one has

$$\begin{aligned}\eta([X,Y]) - \operatorname{Lie}_X(\eta(Y)) + \operatorname{Lie}_Y(\eta(X)) = \\ = \operatorname{Lie}_{[X,Y]}(f) - \operatorname{Lie}_X \operatorname{Lie}_Y(f) + \operatorname{Lie}_Y \operatorname{Lie}_X(f) = 0\end{aligned}$$

hence $d_1(\alpha) = 0$ on all closed forms. A map $\delta: \Lambda^1(M) \longrightarrow \Lambda^2(M)$ which vanishes on closed forms and satisfies the Leibniz rule is de Rham differential, which can be seen from the axiomatic definition of $d$. ■

## Torsion

**DEFINITION:** Let $\nabla$ be a connection on $\Lambda^1 M$,

$$\Lambda^1 \xrightarrow{\nabla} \Lambda^1 M \otimes \Lambda^1 M.$$

**Torsion of $\nabla$** $T_\nabla : \Lambda^1 M \longrightarrow \Lambda^2 M$ is a map $\nabla \circ \operatorname{Alt} - d$, where $\operatorname{Alt} : \Lambda^1 M \otimes \Lambda^1 M \longrightarrow \Lambda^2 M$ is exterior multiplication.

**REMARK:**

$$\begin{aligned} T_\nabla(f\eta) &= \operatorname{Alt}(f\nabla\eta + df \otimes \eta) - d(f\eta) \\ &= f\Big[\operatorname{Alt}(\nabla\eta) - d\eta\Big] + df \wedge \eta - df \wedge \eta = fT_\nabla(\eta). \end{aligned}$$

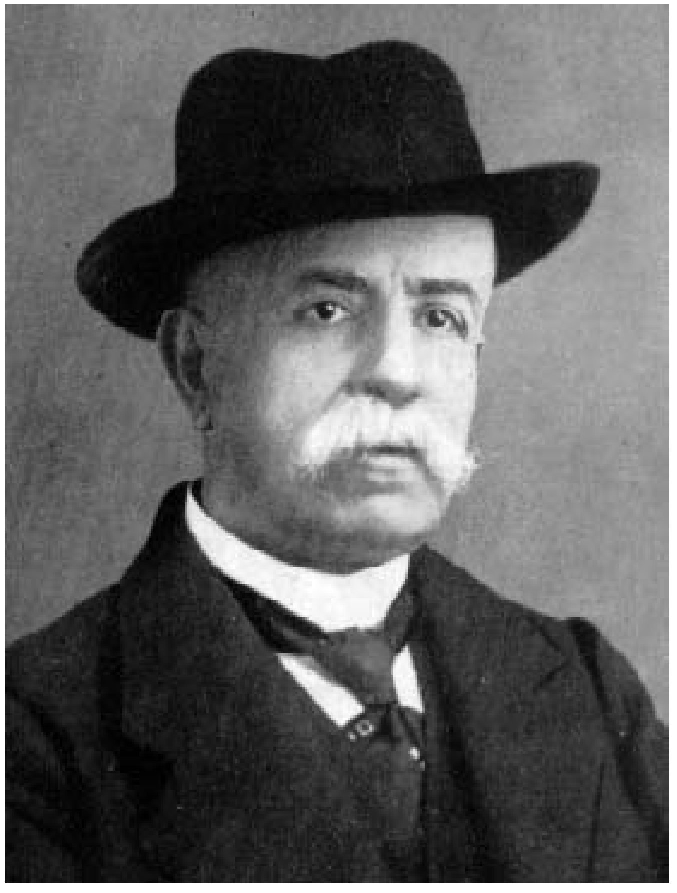
**Therefore $T_\nabla$ is linear**.

**DEFINITION:** Let $(M, g)$ be a Riemannian manifold. A connection $\nabla$ on $TM$ is called **orthogonal** if $\nabla(g) = 0$, and **Levi-Civita connection** if it is orthogonal and has zero torsion.

**THEOREM: ("the fundamental theorem of Riemannian geometry")** **Every Riemannian manifold admits a Levi-Civita connection, and it is unique.**

**Will be proven later today**.

## Gregorio Ricci-Curbastro, Tullio Levi-Civita



Gregorio Ricci-Curbastro,
1853-1925

Tullio Levi-Civita,
1873-1941

*...With his former student Tullio Levi-Civita, he wrote his most famous single publication, a pioneering work on the calculus of tensors, signing it as Gregorio Ricci. This appears to be the only time that Ricci-Curbastro used the shortened form of his name in a publication, and continues to cause confusion.*

## Torsion and commutator of vector fields

**REMARK:** Cartan formula gives

$$\begin{aligned} T_\nabla(\eta)(X,Y) &= \nabla_X(\eta)(Y) - \nabla_Y(\eta)(X) - d\eta(X,Y) \\ &= \nabla_X(\eta)(Y) - \nabla_Y(\eta)(X) - \eta([X,Y]) - \mathrm{Lie}_X(\eta(Y)) + \mathrm{Lie}_Y(\eta(X)). \end{aligned}$$

On the other hand, $\nabla_X(\eta)(Y) = \mathrm{Lie}_X(\eta(Y)) - \eta(\nabla_X(Y))$. Comparing the equations, we obtain

$$T_\nabla(\eta)(X,Y) = \eta\Big(\nabla_X(Y) - \nabla_Y(X) - [X,Y]\Big).$$

**Torsion is often defined as a map $\Lambda^2 TM \longrightarrow TM$ using the formula $\nabla_X(Y) - \nabla_Y(X) - [X,Y]$.**

We have just proved

**CLAIM: The torsion tensor $\nabla_X(Y) - \nabla_Y(X) - [X,Y]$ is dual to the torsion $\nabla \circ \mathrm{Alt} - d : \Lambda^1 M \longrightarrow \Lambda^2 M$ defined above.** ■

## Linearization of the torsion

**REMARK:** Consider the space $\mathcal{A}(\Lambda^1 M)$ of connections on $\Lambda^1 M$. The torsion defines an affine map

$$\mathcal{A}(\Lambda^1 M) \longrightarrow \operatorname{Hom}(\Lambda^1 M, \Lambda^2 M) = TM \otimes \Lambda^2 M.$$

because $T(\nabla + \alpha) = T(\nabla) + \operatorname{Alt}_{12}(\alpha)$, where $\operatorname{Alt}_{12}: \Lambda^1 M \otimes \operatorname{End}(\Lambda^1 M) \longrightarrow \Lambda^2 M \otimes TM$ is antisymmetrization in the first two indices.

**DEFINITION:** **Liearized torsion** is a map

$$T_{lin}: \Lambda^1(M) \otimes \Lambda^1(M) \otimes TM \longrightarrow \Lambda^2 M \otimes TM$$

obtained as a linearization of the torsion map. **It is equal to** $\operatorname{Alt}_{12}$**.**

## Existence of orthogonal connections

**CLAIM:** Let $B$ be a vector bundle equipped with a scalar product. **Then $B$ admits an orthogonal connection**.

**Proof:** Chose a covering $\{U_i\}$, such that $B$ is trivial on each $U_i$ and admits an orthonormal basis in each $U_i$. On each $U_i$ we chose a connection $\nabla_i$ preserving this basis. Let $\psi_i$ be a partition of unit subjugated to $\{U_i\}$. Then **the formula $\nabla(b) := \sum \nabla_i(\psi_i b)$ defines an orthogonal connection**.

**THEOREM: ("the fundamental theorem of Riemannian geometry") Every Riemannian manifold admits a Levi-Civita connection, and it is unique.**

**Proof:** See the next slide.

## Levi-Civita connection (existence and uniqueness)

**Proof. Step 1:** Chose an orthogonal connection $\nabla_0$ on $\Lambda^1 M$. The space $\mathcal{A}$ of orthogonal connections is affine and **its linearization is $\Lambda^1 M \otimes \mathfrak{so}(TM)$.** We shall identify $\mathfrak{so}(TM)$ and $\Lambda^2 M$. Then **$\mathcal{A}$ is an affine space over $\Lambda^1 M \otimes \Lambda^2 M$.**

**Step 2:** Then the linearized torsion map is

$$T_{lin}: \Lambda^1 M \otimes \mathfrak{so}(TM) = \Lambda^1(M) \otimes \Lambda^2 M \xrightarrow{\mathsf{Alt}_{12}} \Lambda^2 M \otimes \Lambda^1 M = \Lambda^2 M \otimes TM.$$

**It is an isomorphism**. Indeed, on the right and on the left there are bundles of the same rank, hence it would suffice to show that $T_{lin} = \mathsf{Alt}_{12}$ is injective. However, if $\eta \in \ker T_{lin}$, it is a form which is symmetric on first two arguments and antisymmetric on the second two, giving $\eta(x,y,z) = \eta(y,x,z) = -\eta(y,z,x)$. This gives $\sigma(\eta) = -\eta$, where $\sigma$ is a cyclic permutation of the arguments. Since $\sigma^3 = 1$, this implies $\eta = 0$.

**Step 3:** We have shown that **an orthogonal connection is uniquely determined by its torsion**. Indeed, torsion map is an isomorphism of affine spaces.

**Step 4:** Let $\nabla := \nabla_0 - T_{lin}^{-1}(T_{\nabla_0})$. Then $T_\nabla = T_{\nabla_0} - T_{lin}(T_{lin}^{-1}(T_{\nabla_0})) = 0$, hence **$\nabla$ is torsion-free.** ■