Hodge theory

lecture 13: Bismut connection

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Connections (reminder)

DEFINITION: Recall that **a connection** on a bundle B is an operator $\nabla : B \longrightarrow B \otimes \Lambda^1 M$ satisfying $\nabla(fb) = b \otimes df + f\nabla(b)$, where $f \longrightarrow df$ is de Rham differential. When X is a vector field, we denote by $\nabla_X(b) \in B$ the term $\langle \nabla(b), X\rangle$.

REMARK: A connection ∇ on B gives a connection $B^* \stackrel{\nabla^*}{\longrightarrow} \Lambda^1 M \otimes B^*$ on the dual bundle, by the formula

$$d(\langle b, \beta\rangle) = \langle \nabla b, \beta\rangle + \langle b, \nabla^* \beta\rangle$$

These connections are usually denoted **by the same letter ∇.**

REMARK: For any tensor bundle $\mathcal{B}_1 := B^* \otimes B^* \otimes ... \otimes B^* \otimes B \otimes B \otimes ... \otimes B$ **a connection on B defines a connection on $\mathcal{B}_1$** using the Leibniz formula:

$$\nabla(b_1 \otimes b_2) = \nabla(b_1) \otimes b_2 + b_1 \otimes \nabla(b_2).$$

Parallel transport along the connection (reminder)

THEOREM: Let B be a vector bundle with connection over $\mathbb{R}$. Then for each $x \in \mathbb{R}$ and each vector $b_x \in B|_x$ **there exists a unique section $b \in B$ such that $\nabla b = 0$, $b|_x = b_x$.**

Proof: This is existence and uniqueness of solutions of an ODE $\frac{db}{dt} + A(b) = 0$. ■

DEFINITION: Let $\gamma : [0,1] \longrightarrow M$ be a smooth path in M connecting x and y, and (B, ∇) a vector bundle with connection. Restricting (B, ∇) to $\gamma([0,1])$, we obtain a bundle with connection on an interval. Solve an equation $\nabla(b) = 0$ for $b \in B\Big|_{\gamma([0,1])}$ and initial condition $b|_x = b_x$. This process is called **parallel transport** along the path via the connection. The vector $b_y := b|_y$ is called **vector obtained by parallel transport of b_x along γ. Holonomy group** of γ is the group of endomorphisms of the fiber B_x obtained from parallel transports along all paths starting and ending in $x \in M$

Lie algebra and tensors (reminder)

DEFINITION: Let V be a representation of a Lie algebra $\mathfrak{g}$. **Then V^* is also a representation;** the action of $\mathfrak{g}$ on V^* is given by the formula $\langle g(x), \lambda\rangle = -\langle x, g(\lambda)\rangle$, for all $x \in V, \lambda \in V^*$. A tensor product of two $\mathfrak{g}$-representations V_1, V_2 is also a $\mathfrak{g}$-representation, with the action of $\mathfrak{g}$ defined by $g(x\otimes y) = g(x)\otimes y + x\otimes g(y)$. This defines the action of $\mathfrak{g}$ on all tensor powers $V^{\otimes i}\otimes (V^*)^{\otimes j}$, which are called **the tensor representations** of $\mathfrak{g}$. We say that $\mathfrak{g}$ **preserves a tensor Φ** if $g(\Psi) = 0$ for all $g \in \mathfrak{g}$.

EXAMPLE: The algebra of all $g \in \operatorname{End}(V)$ preserving a non-degenerate bilinear symmetric form $h \in \operatorname{Sym}^2(V^*)$ is called **orthogonal algebra**, denoted $\mathfrak{so}(V, h)$ or $\mathfrak{so}(V)$. Since $g \in \mathfrak{so}(V)$ if and only if $h(g(x), y) = -h(x, g(y))$, **$\mathfrak{so}(V)$ is represented by antisymmetric matrices.**

CLAIM: Let $h \in \operatorname{Sym}^2(V^*)$ be a non-degenerate bilinear symmetric form. Using h, we identify V and V^*. This gives an isomorphism $V^*\otimes V^* \xrightarrow{\tau} V^*\otimes V = \operatorname{End}(V)$. **Then $\tau(\Lambda^2 V^*) = \mathfrak{so}(V)$.**

Proof: For any $f \in \operatorname{End}(V)$, the 2-form $\tau^{-1}(f)$ is written as $x, y \longrightarrow h(f(x), y)$. By definition, $f \in \mathfrak{so}(V)$ means that $h(f(x), y) = -h(x, f(y))$ and this happens if and only if $\tau^{-1}(f)$ is antisymmetric. ■

The Lie algebra $\mathfrak{u}(V)$ (reminder)

EXAMPLE: Let (V, I) be a real vector space with a complex structure map $I: V \longrightarrow V$, $I^2 = -\operatorname{Id}$, and a Hermitian (that is, I-invariant) scalar product. Define **the unitary Lie algebra** $\mathfrak{u}(V) = \{f \in \operatorname{End}(V) \ \ | \ \ f(I) = f(h) = 0\}$. This is the same as the space of I-invariant orthogonal matrices.

CLAIM: Consider the natural map $V^* \otimes V^* \stackrel{\tau}{\longrightarrow} V^* \otimes V = \operatorname{End}(V)$ associated with h. **Then $\tau(\Lambda^{1,1}(V^*)) = \mathfrak{u}(V)$.**

Proof: The isomorphism τ is I-invariant, because h is I-invariant. **Then $\tau^{-1}(\mathfrak{u}(V))$ is the space of I-invariant 2-forms,** which is precisely $\Lambda^{1,1}(V^*)$.
■

Affine space of orthogonal connections (reminder)

CLAIM: Let B be a bundle with a scalar product. Then **the space of orthogonal connections on B an affine space over $\Lambda^1 M \otimes \mathfrak{so}(B)$.**

Proof: Let $s \in B^* \otimes B^*$ be a 2-form on B. The action of $A := \nabla - \nabla_1$ on $B^* \otimes B^*$ is given by $A(s)(x,y) = -s(A(x),y) - s(x,A(y))$. Therefore, a difference A of orthogonal connections satisfies $h(A(x),y) = -h(x,A(y))$ for all $x,y \in B$. This is the same as $A \in \Lambda^1 M \otimes \mathfrak{so}(B)$. ■

Similarly one proves

CLAIM: Let B be a bundle with a Hermitian structure product. Then **the space of orthogonal connections on B an affine space over $\Lambda^1 M \otimes \mathfrak{u}(B)$.**

CLAIM: Let B be a bundle with a Hermitian structure and a tensor Φ, and $\mathfrak{g} \subset \operatorname{End}(B)$ the Lie algebra of endomorphisms preserving Φ. Then **the space of connections on B preserving Φ is an affine space over $\Lambda^1 M \otimes \mathfrak{g}$.**

Torsion (reminder)

DEFINITION: Let ∇ be a connection on $\Lambda^1 M$,

$$\Lambda^1 \xrightarrow{\nabla} \Lambda^1 M \otimes \Lambda^1 M.$$

Torsion of ∇ $T_\nabla : \ \Lambda^1 M \longrightarrow \Lambda^2 M$ is a map $\nabla \circ \mathrm{Alt} - d$, where $\mathrm{Alt} : \ \Lambda^1 M \otimes \Lambda^1 M \longrightarrow \Lambda^2 M$ is exterior multiplication.

REMARK: Torsion is often defined as a map $\Lambda^2 TM \longrightarrow TM$ using the formula $\nabla_X(Y) - \nabla_Y(X) - [X,Y]$. This map coincides with the torsion map $\Lambda^1 M \longrightarrow \Lambda^2 M$ defined above.

DEFINITION: Let (M,g) be a Riemannian manifold. A connection ∇ on TM is called **orthogonal** if $\nabla(g) = 0$, and **Levi-Civita connection** if it is orthogonal and has zero torsion.

THEOREM: ("the fundamental theorem of Riemannian geometry") Every Riemannian manifold admits a Levi-Civita connection, and it is unique.

Levi-Civita connection, its existence and uniqueness (reminder)

Proof. Step 1: Chose an orthogonal connection ∇_0 on $\Lambda^1 M$. The space $\mathcal{A}$ of orthogonal connections is affine and **its linearization is $\Lambda^1 M \otimes \mathfrak{so}(TM)$.** We shall identify $\mathfrak{so}(TM)$ and $\Lambda^2 M$. Then **$\mathcal{A}$ is an affine space over $\Lambda^1 M \otimes \Lambda^2 M$.**

Step 2: Then the linearized torsion map is

$$T_{lin}:\ \Lambda^1 M \otimes \mathfrak{so}(TM) = \Lambda^1(M) \otimes \Lambda^2 M \stackrel{\mathsf{Alt}_{12}}{\longrightarrow} \Lambda^2 M \otimes \Lambda^1 M = \Lambda^2 M \otimes TM.$$

It is an isomorphism. Indeed, on the right and on the left there are bundles of the same rank, hence it would suffice to show that $T_{lin} = \mathsf{Alt}_{12}$ is injective. However, if $\eta \in \ker T_{lin}$, it is a form which is symmetric on first two arguments and antisymmetric on the second two, giving $\eta(x,y,z) = \eta(y,x,z) = -\eta(y,z,x)$. This gives $\sigma(\eta) = -\eta$, where σ is a cyclic permutation of the arguments. Since $\sigma^3 = 1$, this implies $\eta = 0$.

Step 3: We have shown that **an orthogonal connection is uniquely determined by its torsion**. Indeed, torsion map is an isomorphism of affine spaces.

Step 4: Let $\nabla := \nabla_0 - T_{lin}^{-1}(T_{\nabla_0})$. Then $T_\nabla = T_{\nabla_0} - T_{lin}(T_{lin}^{-1}(T_{\nabla_0})) = 0$, hence **$\nabla$ is torsion-free.** ■

Space of Cartan tensors

DEFINITION: Let $C(V) \subset V \otimes V \otimes V$ be $\ker \mathrm{Sym} \cap \ker \mathrm{Alt}$, where Sym is the symmetrization map $\mathrm{Sym}: V \otimes V \otimes V \longrightarrow \mathrm{Sym}^3(V)$ and Alt the antisymmetrization map $\mathrm{Alt}: V \otimes V \otimes V \longrightarrow \Lambda^3(V)$. Then $C(V)$ is called **the space of Cartan tensors on V**.

REMARK: Clearly, there is a direct sum decomposition $V \otimes V \otimes V = \Lambda^3(V) \oplus \mathrm{Sym}^3(V) \oplus C(V)$.

Lemma 1: Denote by Sym_{ij}, Alt_{ij} the operators of symmetrization and antisymmetrization of $\Phi \in V^{\otimes 3}$ using the indices i, j. **Then**

$$C(V) = \mathrm{Alt}_{12}(\mathrm{Sym}_{23}(V^{\otimes 3})) \oplus \mathrm{Sym}_{12}(\mathrm{Alt}_{23}(V^{\otimes 3})).$$

Proof: Since $\mathrm{Sym}_{23}(V^{\otimes 3})$ is generated by $x \otimes y \otimes y$, one has $\mathrm{im}(\mathrm{Alt}_{12}\mathrm{Sym}_{23}) \supset C(V)$. Similarly, $\mathrm{im}(\mathrm{Alt}_{12}\mathrm{Sym}_{23}) \supset C(V)$.

For a converse statement, we use the decomposition $V \otimes V = \mathrm{Sym}^2 V \oplus \Lambda^2 V$. This gives

$$V^{\otimes 3} = \mathrm{im}\,\mathrm{Alt}_{12}\mathrm{Sym}_{23} \oplus \mathrm{im}\,\mathrm{Alt}_{12}\mathrm{Alt}_{23} \oplus \mathrm{im}\,\mathrm{Sym}_{12}\mathrm{Alt}_{23} \oplus \mathrm{im}\,\mathrm{Sym}_{12}\mathrm{Sym}_{23}.$$

Then $\ker \mathrm{Sym} \cap \ker \mathrm{Alt}$ is precisely $\mathrm{im}\,\mathrm{Alt}_{12}\mathrm{Sym}_{23} \oplus \mathrm{im}\,\mathrm{Sym}_{12}\mathrm{Alt}_{23}$. ■

Torsion and the differential forms

DEFINITION: When $B = \Lambda^1 M$, consider the exterior multiplication map $\operatorname{Alt}: \Lambda^i M \otimes \Lambda^1 M \longrightarrow \Lambda^{i+1} M$. Define **the torsion map** $T_\nabla(\eta) := \operatorname{Alt}(\nabla(\eta)) - d\eta$. Then T_∇ is equal to torsion on $\Lambda^1 M$ and satisfies the Leibnitz identity, which can be used to extend T_∇ from $\Lambda^1 M$ to $\Lambda^* M$:

$$T_\nabla(\lambda \wedge \mu) = T_\nabla(\lambda) \wedge \mu + (-1)^{\tilde{\lambda}} \lambda \wedge T_\nabla(\mu)$$

Symplectic connections

DEFINITION: An almost symplectic structure on a manifold is a non-degenerate 2-form.

EXERCISE: Let (M, ω) be an almost symplectic manifold. Prove that there exists a connection ∇ on TM such that $\nabla(\omega) = 0$. We call such connection **a symplectic connection**.

Lemma 2: Let $\omega \in \Lambda^2 M$ be an almost symplectic structure, and ∇ a symplectic connection. Using ω, we will identify TM and $\Lambda^1 M$, and then we can consider the torsion tensor T_∇ of ∇ as a section $\tau \in \Lambda^2 M \otimes \Lambda^1 M$. Let $\rho := \operatorname{Alt}(T_\nabla)$. **Then $d\omega = 2\rho$.**

Proof: $T_\nabla(\omega) = d\omega$, because $\nabla(\omega) = 0$. However, $T_\nabla(\omega) = \operatorname{Alt}(A_1(\omega \otimes T_\nabla) - A_2(\omega \otimes T_\nabla))$, where $A_i : \Lambda^2 M \otimes TM \otimes \Lambda^2 M$ is the convolution of i-th component of $\omega \otimes T_\nabla$ and the last. Clearly, $A_i(\omega \otimes T_\nabla) = \tau$. This gives $T_\nabla(\omega) = d\omega = 2\rho$. ■

Torsion of almost symplectic structures

Theorem 1: Let (M,ω) be an almost symplectic manifold, and ∇ a symplectic connection. Denote its torsion by $T_\nabla \in \Lambda^2 M \otimes TM$. Using the form ω, we identify TM and $\Lambda^1 M$ and consider T_∇ as a section $\tau \in \Lambda^2 M \otimes \Lambda^1 M$. Denote by Alt_{123} the multiplication map $\Lambda^2 M \otimes \Lambda^1 M \longrightarrow \Lambda^3 M$. **Then $\operatorname{Alt}_{123}(\tau) = \frac{1}{2} d\omega$.** Moreover, **any tensor $\mathfrak{T} \in \Lambda^2 M \otimes \Lambda^1 M$ such that $\operatorname{Alt}_{123}(\tau) = \frac{1}{2} d\omega$ can be realized as a torsion of a symplectic connection.**

Proof. Step 1: Let $\mathfrak{sp}(TM)$ be the Lie algebra of all tensors $a \in \operatorname{End}(TM)$ such that $\omega(a(x), y) = -\omega(x, a(y))$. The same argument as the one used to show $\mathfrak{so}(TM) = \Lambda^2 M$ shows that $\mathfrak{sp}(TM) = \operatorname{Sym}^2(\Lambda^1 M)$.

Step 2: The space of symplectic connections is an affine space with linearization $\Lambda^1 M \otimes \mathfrak{sp}(TM) = \Lambda^1 M \otimes \operatorname{Sym}^2(\Lambda^1 M)$. The image of the linearized torsion map $T_{lin} = \operatorname{Alt}_{12}$ belongs to $C(V)$ (Lemma 1). Therefore, the image of $\operatorname{Alt}_{123}(\tau)$ is independent from the choice of ∇. Any tensor $\mathfrak{T} \in \Lambda^2 M \otimes TM$ with $\operatorname{Alt}_{123}(\tau) = \operatorname{Alt}_{123}(\mathfrak{T})$ can be obtained as a torsion of an appropriate connection, because the part of $C(V)$ which is antisymmetric in the first two multipliers is precisely $\operatorname{Alt}_{12}(\Lambda^1 M \otimes \operatorname{Sym}^2(\Lambda^1 M))$.

Step 3: $\operatorname{Alt}_{123}(\tau) = \frac{1}{2} d\omega$ (Lemma 2). ■

Torsion of Hermitian connection

PROPOSITION: Let (M,I,ω) be an Hermitian complex manifold, ∇ a connection on TM preserving I and ω, and $T_\nabla \in \Lambda^2 M \otimes TM = \Lambda^2 M \otimes \Lambda^1 M$ (we identify TM and $\Lambda^1 M$ using the Riemannian structure). **Then**

$$T_\nabla \in \left(\Lambda^{2,0}(M) \otimes \Lambda^{0,1}(M)\right) \oplus \left(\Lambda^{0,2} \otimes \Lambda^{1,0}(M)\right) \oplus \left(\Lambda^{1,1}(M) \otimes \Lambda^1 M\right). \qquad (**)$$

Proof. Step 1: Integrability of I implies that $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$. Since $\nabla(I) = 0$, one also has $\nabla_X(T^{1,0}M) \subset T^{1,0}M$ for any vector field $X \in TM$. This gives $\nabla_X(Y) - \nabla_Y(X) - [X,Y] \in T^{1,0}M$ for any $X, Y \in T^{1,0}M$. We have shown that

$$T_\nabla \in \left(\Lambda^{2,0}(M) \otimes T^{1,0}(M)\right) \oplus \left(\Lambda^{0,2} \otimes T^{0,1}(M)\right) \oplus \left(\Lambda^{1,1}(M) \otimes \Lambda^1 M\right).$$

Step 2: Since the Riemannian form g is of type (1,1), it pairs (0,1)-vectors and (1,0)-vectors. Therefore, it identifies $T^{1,0}M$ with $\Lambda^{0,1}(M)$. This proves (**). ■

Jean-Michel Bismut (born 26 February 1948)

Bismut connection

THEOREM: (Bismut) Let (M, I, ω) be an Hermitian complex manifold. Then there exists a unique connection ∇ preserving I and ω, such that its torsion $T_\nabla \in \Lambda^2 M \otimes TM = \Lambda^2 M \otimes \Lambda^1 M$ (we identify TM and $\Lambda^1 M$ using the Riemannian metric) **is antisymmetric:** $T_\nabla \in \Lambda^3 M \subset \Lambda^2 M \otimes \Lambda^1 M$. Moreover, **in this case** $T_\nabla = -\frac{1}{2} I(d\omega)$.

REMARK: This connection is called **the Bismut connection**. When (M, I, ω) is Kähler, it is torsion-free and orthogonal, hence ∇ **is the Levi-Civita connection.** We obtain that **on a Kähler manifold, Levi-Civita connection satisfies** $\nabla(I) = 0$.

Proof. Step 1: There are two different ways to identify $\Lambda^2 M \otimes TM$ and $\Lambda^2 M \otimes \Lambda^1 M$: using $g: TM \xrightarrow{\sim} \Lambda^1 M$ and using $\omega: TM \xrightarrow{\sim} \Lambda^1 M$. Denote the first tensor by τ_g and the second by τ_ω. It is clear that $I_3(\tau_g) = \tau_\omega$, where $I_3(x \otimes y \otimes z) = x \otimes y \otimes I(z)$. Torsion of symplectic connections was described earlier today (Theorem 1): we have shown that $\operatorname{Alt}(\tau_\omega) = \frac{1}{2} d\omega$. **This implies that the image of the linearized torsion** $T_{lin}(\Lambda^1 M \otimes \mathfrak{u}(TM))$ **satisfies** $\operatorname{Alt}(I_3(T_{lin}(\Lambda^1 M \otimes \mathfrak{u}(TM))) = 0$.

Bismut connection (2)

Proof. Step 1: The image of the linearized torsion $T_{lin}(\Lambda^1 M \otimes \mathfrak{u}(TM))$ **satisfies** $\operatorname{Alt}(I_3(T_{lin}(\Lambda^1 M \otimes \mathfrak{u}(TM))) = 0$.

Step 2: The torsion of ∇ belongs to the space

$$\mathfrak{W} := \left(\Lambda^{2,0}(M) \otimes \Lambda^{0,1}(M)\right) \oplus \left(\Lambda^{0,2} \otimes \Lambda^{1,0}(M)\right) \oplus \left(\Lambda^{1,1}(M) \otimes \Lambda^1 M\right),$$

as shown above. The linearized torsion map is $T_{lin}: \ \Lambda^1 M \otimes \mathfrak{u}(TM) \longrightarrow \mathfrak{W}$. By the same argument as in the proof of existence of Levi-Civita connection, this map is injective. **This gives an exact sequence**

$$0 \longrightarrow \Lambda^1 M \otimes \mathfrak{u}(TM) \xrightarrow{T_{lin}} \mathfrak{W} \xrightarrow{I_3 \circ \operatorname{Alt}} \Lambda^{2,1}(M) \oplus \Lambda^{1,2}(M) \longrightarrow 0, \quad (***)$$

The last arrow of (***) is surjective because any (2,1)+(1,2)-form can be obtained as anti-symmetrization of $\alpha \in I_3(\mathfrak{W})$. The sequence (***) is exact in the middle term because dimension of the middle term is equal to sum of dimensions of the left and right terms.

Bismut connection (3)

Step 2: Let $\mathfrak{W} := (\Lambda^{2,0}(M)\otimes\Lambda^{0,1}(M))\oplus(\Lambda^{0,2}\otimes\Lambda^{1,0}(M))\oplus(\Lambda^{1,1}(M)\otimes\Lambda^1 M)$. Then **the sequence**

$$0 \longrightarrow \Lambda^1 M\otimes\mathfrak{u}(TM) \xrightarrow{T_{lin}} \mathfrak{W} \xrightarrow{I_3\circ \mathsf{Alt}} \Lambda^{2,1}(M)\oplus\Lambda^{1,2}(M)\longrightarrow 0 \quad (***)$$

is exact.

Step 3: Let $\mathfrak{U}\subset\mathfrak{W}$ be a subspace consisting of all antisymmetric 3-forms, $\mathfrak{U} = \Lambda^{2,1}(M)\oplus\Lambda^{1,2}(M)$. Clearly, for any differential form η, one has $\mathsf{Alt}(I_3(\eta)) = W(\eta)$, where W is **the Weil operator** acting as $W(\eta)(x,y,z) = \eta(Ix,y,z)+\eta(x,Iy,z)+\eta(x,y,Iz)$. Then $\mathfrak{U}\xrightarrow{I_3\circ \mathsf{Alt}}\Lambda^{2,1}(M)\oplus\Lambda^{1,2}(M)$ is bijective. Therefore, **there exists a unique form $\sigma\in\mathfrak{U}$ such that $\mathsf{Alt}(I_3(\sigma)) = \frac{1}{2}d\omega$.**

Step 4: Let ∇_0 be a connection on TM which satisfies $\nabla_0(g)=\nabla_0(I)=0$ **(prove that it exists)**, and $\tau_g\in\mathfrak{W}$ its torsion. Then $\mathsf{Alt}(I_3(\tau_g)) = \mathsf{Alt}(I_3(\sigma)) = \frac{1}{2}d\omega$ by Theorem 1. Therefore, **there exists a unique $A\in\Lambda^1 M\otimes\mathfrak{u}(TM)$ such that $T_{lin}(A)+\tau_g=\sigma$, and the torsion of connection $\nabla := \nabla_0 + A$ is equal to σ.**

Step 5: Step 3 gives $\sigma = \frac{1}{2}W^{-1}(d\omega)$. However, $d\omega$ is (2,1)+(1,2)-form, and for such forms $W=I$, hence $\sigma = -\frac{1}{2}I(d\omega)$. ■