Hodge theory

lecture 14: Supersymmetry for Kähler manbifolds (1)

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Hodge $*$ operator (reminder)

Let V be a vector space. **A metric g on V induces a natural metric on each of its tensor spaces:** $g(x_1 \otimes x_2 \otimes ... \otimes x_k, x'_1 \otimes x'_2 \otimes ... \otimes x'_k) = g(x_1, x'_1) g(x_2, x'_2) ... g(x_k, x'_k)$.

This gives a natural positive definite scalar product on differential forms over a Riemannian manifold (M, g): $g(\alpha, \beta) := \int_M g(\alpha, \beta) \operatorname{Vol}_M$

Another non-degenerate form is provided by the **Poincare pairing:** $\alpha, \beta \longrightarrow \int_M \alpha \wedge \beta$.

DEFINITION: Let M be a Riemannian n-manifold. Define **the Hodge $*$ operator** $* : \Lambda^k M \longrightarrow \Lambda^{n-k} M$ by the following relation: $g(\alpha, \beta) = \int_M \alpha \wedge * \beta$.

REMARK: The Hodge $*$ operator always exists. It is defined explicitly in an orthonormal basis $\xi_1, ..., \xi_n \in \Lambda^1 M$:

$$*(\xi_{i_1} \wedge \xi_{i_2} \wedge ... \wedge \xi_{i_k}) = (-1)^s \xi_{j_1} \wedge \xi_{j_2} \wedge ... \wedge \xi_{j_{n-k}},$$

where $\xi_{j_1}, \xi_{j_2}, ..., \xi_{j_{n-k}}$ is a complementary set of vectors to $\xi_{i_1}, \xi_{i_2}, ..., \xi_{i_k}$, and s the signature of a permutation $(i_1, ..., i_k, j_1, ..., j_{n-k})$.

REMARK: $*^2\big|_{\Lambda^k(M)} = (-1)^{k(n-k)} \operatorname{Id}_{\Lambda^k(M)}$

$d^* = (-1)^{nk} * d*$ **(reminder)**

CLAIM: On a compact Riemannian n-manifold, one has $d^*\big|_{\Lambda^k M} = (-1)^{nk} * d *$, where d^* denotes **the adjoint operator**, which is defined by the equation $(d\alpha, \gamma) = (\alpha, d^*\gamma)$.

Proof: Since

$$0 = \int_M d(\alpha \wedge \beta) = \int_M d(\alpha) \wedge \beta + (-1)^{\tilde\alpha} \alpha \wedge d(\beta),$$

one has $(d\alpha, *\beta) = (-1)^{\tilde\alpha}(\alpha, *d\beta)$. Setting $\gamma := *\beta$, we obtain

$$(d\alpha, \gamma) = (-1)^{\tilde\alpha}(\alpha, *d(*)^{-1}\gamma) = (-1)^{\tilde\alpha}(-1)^{\tilde\alpha(\tilde n - \tilde\alpha)}(\alpha, *d*\gamma) = (-1)^{\tilde\alpha\tilde n}(\alpha, *d*\gamma).$$

■

REMARK: Since in all applications which we consider, n is even, **I would from now on ignore the sign $(-1)^{nk}$.**

Graded vector spaces and algebras (reminder)

DEFINITION: A **graded vector space** is a space $V^* = \bigoplus_{i\in\mathbb{Z}} V^i$.

REMARK: If V^* is graded, the endomorphisms space $\operatorname{End}(V^*) = \bigoplus_{i\in\mathbb{Z}} \operatorname{End}^i(V^*)$ is also graded, with $\operatorname{End}^i(V^*) = \bigoplus_{j\in\mathbb{Z}} \operatorname{Hom}(V^j, V^{i+j})$

DEFINITION: A **graded algebra**(or "graded associative algebra") is an associative algebra $A^* = \bigoplus_{i\in\mathbb{Z}} A^i$, with the product compatible with the grading: $A^i \cdot A^j \subset A^{i+j}$.

REMARK: A bilinear map of graded paces which satisfies $A^i \cdot A^j \subset A^{i+j}$ is called **graded**, or **compatible with grading**.

REMARK: The category of graded spaces can be defined as a **category of vector spaces with $U(1)$-action,** with the weight decomposition providing the grading. Then **a graded algebra is an associative algebra in the category of spaces with $U(1)$-action**.

DEFINITION: An operator on a graded vector space is called **even** (**odd**) if it shifts the grading by even (odd) number. The **parity** $\tilde{a}$ of an operator a is 0 if it is even, 1 if it is odd. We say that an operator is **pure** if it is even or odd.

Supercommutator (reminder)

DEFINITION: A **supercommutator** of pure operators on a graded vector space is defined by a formula $\{a,b\} = ab - (-1)^{\tilde a \tilde b} ba$.

DEFINITION: A graded associative algebra is called **graded commutative** (or "supercommutative") if its supercommutator vanishes.

EXAMPLE: The Grassmann algebra is supercommutative.

DEFINITION: A graded Lie algebra (Lie superalgebra) is a graded vector space $\mathfrak{g}^*$ equipped with a bilinear graded map $\{\cdot,\cdot\}: \ \mathfrak{g}^* \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ which is graded anticommutative: $\{a,b\} = -(-1)^{\tilde a \tilde b}\{b,a\}$ and satisfies **the super Jacobi identity** $\{c,\{a,b\}\} = \{\{c,a\},b\} + (-1)^{\tilde a \tilde c}\{a,\{c,b\}\}$

EXAMPLE: Consider the algebra $\operatorname{End}(A^*)$ of operators on a graded vector space, with supercommutator as above. **Then $\operatorname{End}(A^*), \{\cdot,\cdot\}$ is a graded Lie algebra.**

Lemma 1: Let d be an odd element of a Lie superalgebra, satisfying $\{d,d\} = 0$, and L an even or odd element. **Then $\{\{L,d\},d\} = 0$.**

Proof: $0 = \{L,\{d,d\}\} = \{\{L,d\},d\} + (-1)^{\tilde L}\{d,\{L,d\}\} = 2\{\{L,d\},d\}$. ∎

Supersymmetry in Kähler geometry

Let (M, I, g) be a Kaehler manifold, ω its Kaehler form. **On $\Lambda^*(M)$, the following operators are defined.**

0. d, d^*, Δ, because it is Riemannian.

1. $L(\alpha) := \omega \wedge \alpha$

2. $\Lambda(\alpha) := *L * \alpha$. It is easily seen that $\Lambda = L^*$.

3. The Weil operator $W\Big|_{\Lambda^{p,q}(M)} = \sqrt{-1}\,(p-q)$

THEOREM: These operators generate a Lie superalgebra $\mathfrak{a}$ of dimension (5|4), acting on $\Lambda^*(M)$. Moreover, the Laplacian Δ is central in $\mathfrak{a}$, hence **$\mathfrak{a}$ also acts on the cohomology of M.**

REMARK: This is a convenient way to summarize the Kähler relations and the Lefschetz' $\mathfrak{sl}(2)$-action.

The coordinate operators

Let V be an even-dimensional real vector space equipped with a scalar product, and $v_1, ..., v_{2n}$ an orthonormal basis. Denote by $e_{v_i} : \Lambda^k V \longrightarrow \Lambda^{k+1} V$ an operator of multiplication, $e_{v_i}(\eta) = v_i \wedge \eta$. Let $i_{v_i} : \Lambda^k V \longrightarrow \Lambda^{k-1} V$ be an adjoint operator, $i_{v_i} = *e_{v_i}*$.

CLAIM: The operators e_{v_i}, i_{v_i}, Id are a basis of an **odd Heisenberg Lie superalgebra** $\mathfrak{H}$, with **the only non-trivial supercommutator given by the formula** $\{e_{v_i}, i_{v_j}\} = \delta_{i,j} \operatorname{Id}$.

Now, consider the tensor $\omega = \sum_{i=1}^n v_{2i-1} \wedge v_{2i}$, and let $L(\alpha) = \omega \wedge \alpha$, and $\Lambda := L^*$ be the corresponding **Hodge operators**.

CLAIM: (Lefschetz triples) From the commutator relations in $\mathfrak{H}$, one obtains immediately that

$$H := [L, \Lambda] = \left[\sum e_{v_{2i-1}} e_{v_{2i}}, \sum i_{v_{2i-1}} i_{v_{2i}}\right] = \sum_{i=1}^{2n} e_{v_i} i_{v_i} - \sum_{i=1}^{2n} i_{v_i} e_{v_i},$$

is a scalar operator acting as $k-n$ on k-forms.

COROLLARY: The triple L, Λ, H satisfies the relations for the $\mathfrak{sl}(2)$ Lie algebra: $[L, \Lambda] = H$, $[H, L] = 2L$, $[H, \Lambda] = 2\Lambda$.

Hodge components of d

CLAIM: Let (M,I) be an almost complex manifold, and $d = \bigoplus d^{i,1-i}$ be the Hodge components of d, with $d^{a,b}: \Lambda^{p,q}(M) \longrightarrow \Lambda^{p+a,q+b}(M)$. **Then there are only 4 components, $d = d^{2,-1} + d^{1,0}, + d^{0,1} + d^{-1,2}$, with $d^{2,-1}$ and $d^{-1,2}$ C^∞-linear.**

Proof. Step 1: Each of the components $d^{i,j}$ satisfies the Leibniz identity. To see this, take the Leibniz identity for d and consider its Hodge components.

Step 2: $d\big|_{\Lambda^1(M)}$ has only 4 Hodge components, because it is mapped to $\Lambda^2(M)$ which has only 4 Hodge components.

Step 3: Since the de Rham algebra is generated by Λ^1, Step 2 implies that all components $d^{i,1-i}$ vanish, except the 4 listed above.

Step 4: $d^{i,1-i}(f\eta) = d^{i,1-i}(f)\wedge\eta + fd^{i,1-i}(\eta)$ for all $f \in C^\infty M$. However, since $\Lambda^1 = \Lambda^{1,0}\oplus\Lambda^{0,1}$, one has $d^{i,1-i}(f) = 0$ for all $i \neq 1, 0$. **Therefore $d^{2,-1}$ and $d^{-1,2}$ are C^∞-linear.** ■

REMARK: The map $d^{-1,2}: \Lambda^{0,1}(M) \longrightarrow \Lambda^{2,0}(M)$, interpreted as an element in $\Lambda^{2,0}\otimes T^{0,1}M$ **is equal to the Nijenhujs tensor.**. Indeed, for any $X, Y \in T^{1,0}(M)$, and any $\eta \in \Lambda^{0,1}(M)$, one has $d\eta(X,Y) = \eta([X,Y]) + \operatorname{Lie}_X \eta(Y) - \operatorname{Lie}_Y \eta(X) = \eta(N(X,Y))$.

The twisted differential d^c

DEFINITION: The **twisted differential** is defined as $d^c := IdI^{-1}$.

CLAIM: Let (M, I) be a complex manifold. **Then $\partial := \frac{d+\sqrt{-1}\, d^c}{2}$, $\overline{\partial} := \frac{d-\sqrt{-1}\, d^c}{2}$ are the Hodge components of d**, $\partial = d^{1,0}$, $\overline{\partial} = d^{0,1}$.

Proof: The Hodge components of d are expressed as $d^{1,0} = \frac{d+\sqrt{-1}\, d^c}{2}$, $d^{0,1} = \frac{d-\sqrt{-1}\, d^c}{2}$. Indeed, $I(\frac{d+\sqrt{-1}\, d^c}{2})I^{-1} = \sqrt{-1}\frac{d+\sqrt{-1}\, d^c}{2}$, hence $\frac{d+\sqrt{-1}\, d^c}{2}$ **has Hodge type (1,0);** the same argument works for $\overline{\partial}$. ∎

CLAIM: On a complex manifold, one has $d^c = [\mathcal{W}, d]$.

Proof: Clearly, $[\mathcal{W}, d^{1,0}] = \sqrt{-1}d^{1,0}$ and $[\mathcal{W}, d^{0,1}] = -\sqrt{-1}d^{0,1}$. Then $[\mathcal{W}, d] = \sqrt{-1}\, d^{1,0} - \sqrt{-1}\, d^{0,1} = IdI^{-1}$. ∎

COROLLARY: $\{d, d^c\} = \{d, \{d, \mathcal{W}\}\} = 0$ (Lemma 1).

Plurilaplacian

THEOREM: Let M, I be a complex manifold. **Then 1. $\partial^2 = 0$.**
2. $\overline{\partial}^2 = 0$.
3. $dd^c = -d^c d$
4. $dd^c = 2\sqrt{-1}\,\partial\overline{\partial}$.

Proof: The first is vanishing of (2,0)-part of d^2, and the second is vanishing of its (0,2)-part. Now, $\{d, d^c\} = -\{d, \{d, \mathcal{W}\}\} = 0$ (Lemma 1), this gives $dd^c = -d^c d$. Finally, $2\sqrt{-1}\partial\overline{\partial} = \frac{1}{2}(d+\sqrt{-1}\,d^c)(d-\sqrt{-1}\,d^c) = \frac{1}{2}(dd^c - d^c d) = dd^c$.
■

DEFINITION: The operator dd^c is called **the pluri-Laplacian**.

EXERCISE: Prove that **on a Riemannian surface (M, I, ω), one has $dd^c(f) = \Delta(f)\omega$.**