Hodge theory

lecture 16: Currents and the Poincaré-Dolbeault-Grothendieck lemma

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Generalized functions

DEFINITION: Let V be a vector space equipped with a collection of norms (or seminorms) $|\cdot|_i$, $i = 0, 1, 2, ...$ and a topology which is given by the metric

$$d(x,y) = \sum_{i=0}^{\infty} 2^{-i} \min(|x-y|_i, 1),$$

assumed to be non-degenerate. The space V is called **a Fréchet space** if this metric is complete.

REMARK: Completeness **is equivalent to convergence of any sequence $\{a_i\}$ which is fundamental with respect to all the (semi-)norms $|\cdot|_i$.**

REMARK: A sequence converges in the Fréchet topology given by d $\Leftrightarrow$ it converges in any of the (semi-)norms $|\cdot|_i$.

DEFINITION: Let M be a Riemannian manifold, and $\nabla^i : C^\infty(M) \longrightarrow \Lambda^1(M)^{\otimes i}$ the iterated connection. **Topology C^k** on the space $C^\infty_c(M)$ of functions with compact support is defined by the norm

$$|\varphi|_{C^k} := \sup_M \sum_{i=0}^{k} |\nabla^i \varphi|.$$

Generalized functions (2)

DEFINITION: The space of test-functions with compact support is the space of functions with compact support and a metric

$$d(x,y) = \sum_{i=0}^{\infty} 2^{-i} \min(|x-y|_{C^i}, 1).$$

of uniform convergence of all derivatives.

EXERCISE: Prove that **the space of test-functions with support in a compact set $K \subset M$ is a Fréchet space.**

DEFINITION: Generalized function (also called **distribution**) is a functional on the space of test-function which is continuous in one of the C^i-topologies on the space $C^\infty(M)_K$ of functions with support in any compact $K \subset M$.

EXAMPLE: Delta-function δ_z is a functional mapping $\varphi \in C^\infty_c(M)$ to $\varphi(z)$, for a given point $z \in M$. **Delta-function is continuous in the topology C^0, its derivative is continuous in C^1 and so on**.

Currents on complex manifolds

REMARK: The C^i**-topology** is defined on the space of sections of any vector bundle B over using the same formula. It depends on the choice of the metric on M and on B, but **the induced topology is clearly independent from this choice.**

DEFINITION: The space of test-forms of type (p,q) on a complex manifold is the space $\Lambda_c^{p,q}(M)$ with compact support, equipped with the Fréchet topology as on the test-functions.

DEFINITION: A (p,q)**-current** on a complex n-dimensional manifold is a functional θ on the space $\Lambda_c^{n-p,n-q}(M)$ of forms with compact support, such that for any compact set $K \subset M$ there exists $i \geqslant 0$ such that θ is continuous in C^i-topology on forms with support in K.

REMARK: A smooth (p,q)-form ψ defines a (p,q)-current: given a test-form $\alpha \in \Lambda_c^{n-p,n-q}(M)$, consider the functional $\alpha \longrightarrow \int_M \psi \wedge \alpha$. This gives an embedding $\Lambda^{p,q}(M) \hookrightarrow \mathcal{D}^{p,q}(M)$ from forms to currents.

REMARK: Currents are (p,q)-forms with coefficients in generalized functions.

Cohomology of currents

DEFINITION: Define **the de Rham differential on the space of currents** using the formula $\langle d\psi, \alpha\rangle := -(-1)^{\tilde\psi}\langle\psi, d\alpha\rangle$. **This definition is compatible with the embedding $\Lambda^{p,q}(M) \hookrightarrow \mathcal{D}^{p,q}(M)$ from forms to currents:**

$$\int_M d\psi\wedge\alpha = \int_M d(\psi\wedge\alpha) - (-1)^{\tilde\psi}\int_M \psi\wedge d\alpha = -(-1)^{\tilde\psi}\int_M \psi\wedge d\alpha$$

by Stokes' formula.

REMARK: **The Dolbeault differentials** $\partial = d^{1,0}$, $\bar\partial = d^{0,1}$ are defined on currents using the same formula.

EXERCISE: **Prove the Poincaré lemma for currents.**

DEFINITION: Let $f: \ X \longrightarrow Y$ be a proper holomorphic map of complex manifolds, $\dim_{\mathbb C} X = \dim_{\mathbb C} Y + k$, and α a (p,q)-current on X. Define **the pushforward** $f_*\alpha$ using $\langle f_*\alpha, \tau\rangle := \langle\alpha, f^*\tau\rangle$, where τ is any test-form. Then **$f_*\alpha$ has bidimension $(p-k, q-k)$.** One should think of f_* as of fiberwise integration.

REMARK: Clearly, $df_*\alpha = f_*d\alpha$, $\partial f_*\alpha = f_*\partial\alpha$, and so on.

REMARK: Pullback of currents is (generally speaking) not well-defined.

Poincaré-Lelong formula

CLAIM: (Poincaré-Lelong formula)
Consider a current on $\mathbb{C}$ given by $\frac{1}{\pi z}dz$. **Then** $d\left(\frac{1}{\pi z}dz\right) = \delta_0 \operatorname{Vol}$, where δ_0 is δ-function in 0.

Proof: For any function smooth f on a closure of a disc D and $w \in D$, Cauchy formula gives

$$f(w) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial D} \frac{f(z)}{z-w} dz - \frac{1}{\pi} \int_D \frac{\overline{\partial} f}{z-w} \wedge dz.$$

Applying this to a test-function f with compact support inside D, we obtain

$$f(w) = -\left\langle \frac{1}{\pi z} dz, \overline{\partial} f \right\rangle = \left\langle \overline{\partial} \left(\frac{1}{\pi z}\right) dz, f \right\rangle = \left\langle d\left(\frac{dz}{\pi z}\right), f \right\rangle.$$

(the last equality is true because $d\eta = \overline{\partial}\eta$ for any $(1,0)$-form on a disc). ■

Poincaré-Dolbeault-Grothendieck (dimension 1)

COROLLARY: Let $\pi_1, \pi_2 : \mathbb{C}^2 \longrightarrow \mathbb{C}$ be coordinate projections, and ξ a (1,0)-current on $\mathbb{C}^2$ defined by $\xi := \frac{1}{\pi(z-w)} dw$, where w, z are coordinates on $\mathbb{C}^2$. Consider **convolution with the current** ξ, given by $P_\xi(\tau) := \pi_{2*}(\pi_1^* \tau \wedge \xi)$. **Then $\overline{\partial} P_\xi(\alpha) = \alpha$ for any $(0,1)$-form α with compact support.**

Proof: $\overline{\partial} P_\xi(\alpha) = \pi_{2*}(\pi_1^* \alpha \wedge \overline{\partial} \xi) = \pi_{2*}(\pi_1^* \alpha \wedge \delta_\triangle) = \alpha$, where $\delta_\triangle$ is δ-function of the diagonal $\triangle$, which is defined as $\langle \kappa, \delta_\triangle \rangle := \int_\triangle \kappa$. ■

COROLLARY: For any (0,1)-form α with compact support on $\mathbb{C}$ there exists a function $f \in C^\infty(\mathbb{C})$ such that $\overline{\partial} f = \alpha$. Moreover, f can be chosen in such a way that $|f(z)| < C \frac{1}{|z|}$ for some constant $C > 0$ depending on $\int_\mathbb{C} |\alpha|$.

Proof: Take $f = P_\xi(\alpha)$. From the definition of P_ξ we obtain $|f(z)| < \operatorname{dist}(z, S)^{-1} \int_\mathbb{C} |\alpha|$, where $S = \operatorname{Supp}(\alpha)$. This implies the estimate. ■

REMARK: Similarly, for any $(1,1)$-form α with compact support one has $\overline{\partial}(P_\xi(\alpha)) = \alpha$, **with the same asymptotic estimates on $P_\xi(\alpha)$.**