# Hodge theory

## lecture 17: Poincaré-Dolbeault-Grothendieck lemma

NRU HSE, Moscow

Misha Verbitsky, March 24, 2018

## Inverting $\overline{\partial}$ using the Hodge theory

**CLAIM:** Let $\beta$ be a $\overline{\partial}$-exact form, and $\gamma := \Delta^{-1}\overline{\partial}^*\beta$. **Then $\overline{\partial}(\gamma) = \beta$.**

**Proof:** Indeed,

$$\overline{\partial}^*\beta = \{\overline{\partial}, \overline{\partial}^*\}(\Delta^{-1}\overline{\partial}^*\beta) = \overline{\partial}^*\overline{\partial}\gamma$$

because $(\overline{\partial}^*)^2 = 0$ and $\Delta^{-1}$ commutes with $\overline{\partial}^*$. However, $\ker \overline{\partial}^*$ is orthogonal to $\operatorname{im} \overline{\partial}$, hence $\overline{\partial}^*\Big|_{\operatorname{im} \overline{\partial}}$ is injective. Then $\overline{\partial}^*\beta = \overline{\partial}^*\overline{\partial}\gamma$ implies $\beta = \overline{\partial}\gamma$. ■

**REMARK:** For a different proof of the following proposition, see Lecture 16.

## Poincaré-Dolbeault-Grothendieck (dimension 1)

**PROPOSITION:** Let $\alpha$ be a (p,1)-form on a disk $D_{r+\varepsilon} \subset \mathbb{C}$ of radius $r+\varepsilon$. **Then $\alpha\big|_{D_r} = P_\xi(\alpha)$, where $P_\xi : \Lambda^{p,1}(D_{r+\varepsilon}) \longrightarrow \Lambda^{p,0}(D_r)$ is an operator which depends only on $r$ and $\varepsilon$.**

**Proof. Step 1:** Hodge theory implies that **cohomology of $\overline{\partial}$ on any Kähler manifold are identified with cohomology of $d$.** Indeed, the corresponding Laplacians coinside.

**Step 2:** Then the cohomology of $\overline{\partial}$ on a 1-dimensional complex torus $T$ are 1-dimensional in each bidegree $(p,q)$. However, averaging of a $(p,q)$-form on a torus gives a map from $\Lambda^{p,q}(T)$ to $T$-invariant forms, which are clearly parallel, hence harmonic. Since torus acts on itself by isometries, this action commutes with harmonic projection. Therefore, **a form is cohomologous to 0 if and only if is average on $T$ vanishes.** This is true for the de Rham differential and for the Dolbeault differential.

**Step 3:** Fix an embedding from $D_{r+\varepsilon}$ to a torus, and let $\psi$ be a cutoff function which is 1 on $D_r$ and 0 outside of $D_{r+\varepsilon}$. Then $\psi\alpha$ is can be extended to a smooth $(p,1)$-form on $T$. Chose a $(p,1)$-form which is supported on $T\backslash D_{r+\varepsilon}$ and has non-zero $\overline{\partial}$-cohomology class $v \in H^{p,1}(T)$ in $T$, and let $A(\psi\alpha) \in H^{p,1}(T)$ be the $\overline{\partial}$-cohomology class of $\psi\alpha$. Denote by $\lambda(\psi\alpha) \in \mathbb{C}$ the number such that $\psi\alpha - \lambda(\psi\alpha)v$ is cohomologous to 0. Then $P_\xi(\alpha) := \Delta^{-1}\overline{\partial}^*(\psi\alpha - \lambda(\psi\alpha)v)$ satisfies $\overline{\partial}(P_\xi(\alpha)) = \alpha + \lambda(\psi\alpha)v$, and this form is equal to $\alpha$ on $D_r \subset T$. ■

## Poincaré-Dolbeault-Grothendieck lemma

**DEFINITION: Polydisc** $D^n$ is a product of $n$ discs $D \subset \mathbb{C}$.

**THEOREM: (Poincaré-Dolbeault-Grothendieck lemma)**
Let $\eta \in \Lambda^{0,p}(D^n)$ be a $\overline{\partial}$-closed form on a polydisc, smoothly extended to a neighbourhood of its closure $\overline{D^n} \subset \mathbb{C}^n$. **Then $\eta$ is $\overline{\partial}$-exact.**

**REMARK:** We have proven PDG-lemma for an $(0,1)$-form $\eta$ with compact support in $\mathbb{C}$. In this case $\eta = \overline{\partial}\alpha$, **where $\alpha$ is a smooth function on $\mathbb{C}$.** This function is not necessarily compactly supported, but **is is bounded by $C/|z|$ for large $z$.**

**REMARK:** Using the decomposition $\Lambda^{p,q}(D^n) \cong \Lambda^{p,0}(D^n) \otimes \Lambda^{0,q}(D^n)$, any form can be represented by a sum $\sum_i \alpha_i^{0,q} \wedge P_i^{p,0}$, where $P_i$ are monomials on $dz_i$, where $z_i$ are holomorphic coordinate functions. Since $\overline{\partial}(\alpha_i^{0,q} \wedge P_i^{p,0}) = \overline{\partial}(\alpha_i^{0,q}) \wedge P_i^{p,0}$, **it suffices to prove the Poincaré-Dolbeault-Grothendieck lemma for $(0,q)$-forms.**

**REMARK:** To prove vanishing of cohomology of $\overline{\partial}: \Lambda^{0,q}(M) \longrightarrow \Lambda^{0,q+1}(M)$, it suffices to construct **the homotopy operator**, that is, a map $\overline{\partial}: \Lambda^{0,q}(M) \longrightarrow \Lambda^{0,q-1}(M)$ satisfying $\{\overline{\partial}, \gamma\} = \mathrm{Id}$. **This is how we prove the Poincaré lemma in Handout 8.**

## Proof of Poincaré-Dolbeault-Grothendieck lemma

**Proof. Step 1:** Let $\overline{\partial}_i : \Lambda^{0,q}(D^n) \longrightarrow \Lambda^{0,q+1}(D^n)$ be the operator $\alpha \longrightarrow d\overline{z}_i \wedge \frac{d}{d\overline{z}_i}\alpha$, where $z_i$ is $i$-th coordinate on $D^n$. **Then $\overline{\partial} = \sum_i \overline{\partial}_i$.**

**Step 2:** By PDG-lemma in dimension 1, cohomology of $\partial_i$ vanish. Denote by $\gamma_i$ the corresponding integral operator $P_\xi$. If $\alpha = d\overline{z}_i \wedge \beta$, one has $\{\overline{\partial}_i, \gamma_i\}(\alpha) = \alpha$. If $\alpha$ contains no monomials divisible by $d\overline{z}_i$, one has $\overline{\partial}_i\{\overline{\partial}_i, \gamma_i\}(\alpha) = 0$. This implies that $\operatorname{im}\left[\{\overline{\partial}_i, \gamma_i\} - \operatorname{Id}\right]$ **lies in the space $R_i$ forms without $d\overline{z}_i$ in monomial decomposition and with all coefficients holomorphic as functions on $z_i$.**

**Step 3:** Properties $\gamma_i$:
**(1).** $\operatorname{im}\left[\{\overline{\partial}_i, \gamma_i\} - \operatorname{Id}\right] \subset R_i$. **(2).** $\{\overline{\partial}_i, \gamma_j\} = 0$, **if** $i \neq j$. **(3).** $\left[\{\overline{\partial}_i, \gamma_i\}\right]\Big|_{R_i} = 0$. **(4).** $\gamma_i(R_j) \subset R_j$, $\overline{\partial}_i(R_j) \subset R_j$ **for all** $i \neq j$.
Property (1) is proven in Step 2, properties (2) and (4) are implied by the exlicit formula for $\gamma_i$. Finally, (3) follows because for all forms $\alpha$ without $d\overline{z}_i$ in monomial decomposition one has $\{\gamma_i, \overline{\partial}\}(\alpha) = \gamma_i(\overline{\partial}_i(\alpha))$.

**Step 4:** Properties (1), (3) and (4) give $\left[\{\overline{\partial}_i, \gamma_i\} - \operatorname{Id}\right](R_{i_1} \cap R_{i_2} \cap ... \cap R_{i_k}) \subset R_i \cap R_{i_1} \cap R_{i_2} \cap ... \cap R_{i_k}$ for $i \notin \{i_1, i_2, ..., i_k\}$, and $\{\overline{\partial}_i, \gamma_i\}\Big|_{R_{i_1} \cap R_{i_2} \cap ... \cap R_{i_k}} = 0$ otherwise.

## Proof of Poincaré-Dolbeault-Grothendieck lemma (2)

**Step 4:** Properties (1), (3) and (4) give $\left[\{\overline{\partial}_i, \gamma_i\} - \operatorname{Id}\right](R_{i_1} \cap R_{i_2} \cap ... \cap R_{i_k}) \subset R_i \cap R_{i_1} \cap R_{i_2} \cap ... \cap R_{i_k}$ for $i \notin \{i_1, i_2, ..., i_k\}$, and $\{\overline{\partial}_i, \gamma_i\}\Big|_{R_{i_1}\cap R_{i_2}\cap ... \cap R_{i_k}} = 0$ otherwise.

**Step 5:** Let $\gamma := \sum_i \gamma_i$. Since $\{\overline{\partial}_i, \gamma_j\} = 0$ for $i \neq j$, Step 4 gives

$$\left[\{\overline{\partial}, \gamma\} - (n-k)\operatorname{Id}\right](R_{i_1} \cap R_{i_2} \cap ... \cap R_{i_k}) \subset \sum_{i \neq i_1, i_2, ..., i_k} R_i \cap R_{i_1} \cap R_{i_2} \cap ... \cap R_{i_k}$$

**Step 6:** Let $W_0$ be the space of $(0,p)$-forms on $D^n$ which can be smoothly extended in a certain neighbourhood of the closure $\overline{D^n} \subset \mathbb{C}^n$, and $W_k \subset W_{k-1}$ its subspace generated by all $R_{i_1} \cap R_{i_2} \cap ... \cap R_{i_k}$ for $i_1 < i_2 < ... < i_k$. **Step 5 implies $\left[\{\overline{\partial}, \gamma\} - (n-k)\operatorname{Id}\right]\Big|_{W_k} \subset W_{k+1}$.**

**Step 7:** Clearly, $W_n = 0$ for any $p > 0$: elements of this space are $(0,p)$-forms without any $d\overline{z}_i$ in its monomial decomposition. Using induction in $d = n-k$, **we can assume that any $\overline{\partial}$-closed form in $W_{k+1}$ is $\overline{\partial}$-closed; to prove PDG-lemma, it would suffice to prove the same for any $\overline{\partial}$-closed form $\alpha \in W_k$.** Step 6 gives $(n-k)\alpha - \{\overline{\partial}, \gamma\}(\alpha) = (n-k)\alpha - \overline{\partial}\gamma(\alpha) \in W_{k+1}$, and this for is $\overline{\partial}$-exact by the induction assumption. **This gives $(n-k)\alpha - \overline{\partial}\gamma(\alpha) = \overline{\partial}\eta$**, hence $\alpha$ is $\overline{\partial}$-exact. ■

## Algebra of supersymmetry of a Kähler manifold: reminder

Let $(M,I,g)$ be a Kaehler manifold, $\omega$ its Kaehler form. **On $\Lambda^*(M)$, the following operators are defined.**

0. $d$, $d^*$, $\Delta$, because it is Riemannian.

1. $L(\alpha) := \omega\wedge\alpha$

2. $\Lambda(\alpha) := *L*\alpha$. It is easily seen that $\Lambda = L^*$.

3. The Weil operator $W\Big|_{\Lambda^{p,q}(M)} = \sqrt{-1}\,(p-q)$

**THEOREM: These operators generate a Lie superalgebra $\mathfrak{a}$ of dimension** $(5|4)$**,** acting on $\Lambda^*(M)$. Moreover, the Laplacian $\Delta$ is central in $\mathfrak{a}$, hence **$\mathfrak{a}$ also acts on the cohomology of $M$.**

The odd part of this algebra generates "odd Heisenberg algebra" $\langle d, d^c, d^*, (d^c)^*, \Delta\rangle$, with the only non-zero anticommutator $\{d,d^*\} = \{d^c,(d^c)^*\} = \Delta$.

The even part of this algebra contains an $\mathfrak{sl}(2)$-triple $\langle L,\Lambda,H\rangle$ acting on $\mathfrak{a}^{\text{odd}}$ as on a direct sum of two weight 1 representations ("Kodaira relations"). The Weil element commutes with $\langle L,\Lambda,H,\Delta\rangle$ and acts on $\mathfrak{a}^{\text{odd}}$ via $[W,d]=d^c$, $[W,d^*]=(d^c)^*$.

## Inverting $\overline{\partial}$ using the Hodge theory (reminder)

**CLAIM:** Let $\beta$ be a $\overline{\partial}$-exact form, and $\gamma := \Delta^{-1}\overline{\partial}^*\beta$. **Then $\overline{\partial}(\gamma) = \beta$.**

**Proof:** Indeed,

$$\overline{\partial}^*\beta = \{\overline{\partial}, \overline{\partial}^*\}(\Delta^{-1}\overline{\partial}^*\beta) = \overline{\partial}^*\overline{\partial}\gamma$$

because $(\overline{\partial}^*)^2 = 0$ and $\Delta^{-1}$ commutes with $\overline{\partial}^*$. However, $\ker\overline{\partial}^*$ is orthogonal to $\operatorname{im}\overline{\partial}$, hence $\overline{\partial}^*\Big|_{\operatorname{im}\overline{\partial}}$ is injective. Then $\overline{\partial}^*\beta = \overline{\partial}^*\overline{\partial}\gamma$ implies $\beta = \overline{\partial}\gamma$. ■

**REMARK:** Similarly, f**or any $d$-exact form $\beta$, one has $\beta = \Delta^{-1}d^*\beta$.**

## $dd^c$-lemma

**THEOREM:** Let $\eta$ be a form on a compact Kähler manifold, satisfying one of the following conditions.
(1). $\eta$ is an exact $(p,q)$-form. (2). $\eta$ is $d$-exact, $d^c$-closed.
(3). $\eta$ is $\partial$-exact, $\overline{\partial}$-closed.
**Then** $\eta \in \operatorname{im} dd^c = \operatorname{im} \partial\overline{\partial}$**.**

**Proof:** Notice immediately that in all three cases $\eta$ **is closed and orthogonal to the kernel of** $\Delta$, hence its cohomology class vanishes.

Since $\eta$ is exact, it lies in the image of $\Delta$. Operator $G_\Delta := \Delta^{-1}$ is defined on $\operatorname{im} \Delta = \ker \Delta^\perp$ and commutes with $d, d^c$.

In case (1), $\eta$ is $d$-exact, and $I(\eta) = \overline{\eta}$ is $d$-closed, hence $\eta$ is $d$-exact, $d^c$-closed like in (2).

Then $\eta = d\alpha$, where $\alpha := G_\Delta d^* \eta$. Since $G_\Delta$ and $d^*$ commute with $d^c$, the form $\alpha$ is $d^c$-closed; since it belongs to $\operatorname{im} \Delta = \operatorname{im} G_\Delta$, it is $d^c$-exact, $\alpha = d^c \beta$ which gives $\eta = dd^c\beta$.

In case (3), we have $\eta = \partial\alpha$, where $\alpha := G_\Delta \partial^* \eta$. Since $G_\Delta$ and $\partial^*$ commute with $\overline{\partial}$, the form $\alpha$ is $\overline{\partial}$-closed; since it belongs to $\operatorname{im} \Delta$, it is $\overline{\partial}$-exact, $\alpha = \overline{\partial}\beta$ which gives $\eta = \partial\overline{\partial}\beta$. ■

## Massey products

Let $a, b, c \in \Lambda^*(M)$ be closed forms on a manifold $M$ with cohomology classes $[a], [b], [c]$ satisfying $[a][b] = [b][c] = 0$, and $\alpha, \gamma \in \Lambda^*(M)$ forms which satisfy $d(\alpha) = a \wedge b$, $d(\gamma) = b \wedge c$. Denote by $L_{[a]}, L_{[c]} : H^*(M) \longrightarrow H^*(M)$ the operation of multiplication by the cohomology classes $[a], [c]$.

**Then $\alpha \wedge c - a \wedge \gamma$ is a closed form, and its cohomology class is well-defined modulo** $\operatorname{im} L_{[a]} + \operatorname{im} L_{[c]}$.

**DEFINITION:** Cohomology class $\alpha \wedge c - a \wedge \gamma$ is called **Massey product of $a, b, c$.**

**PROPOSITION: On a Kähler manifold, Massey products vanish.**

**Proof:** Let $a, b, c$ be harmonic forms of pure Hodge type, that is, of type $(p, q)$ for some $p, q$. Then $ab$ and $bc$ are exact pure forms, hence $ab, bc \in \operatorname{im} dd^c$ by $dd^c$-lemma. This implies that $\alpha := d^* G_\Delta(ab)$ and $\gamma := d^* G_\Delta(bc)$ are $d^c$-exact. Therefore $\mu := \alpha \wedge c - a \wedge \gamma$ is a $d^c$-exact, $d$-closed form. **Applying $dd^c$-lemma again, we obtain that $\mu$ is $dd^c$-exact, hence its cohomology class vanish.**

■

## Hartogs theorem

**THEOREM:** Let $f$ be a holomorphic function on $\mathbb{C}^n\backslash K$, where $K \subset \mathbb{C}^n$ is a compact, and $n > 1$. **Then $f$ can be extended to a holomorphic function on $\mathbb{C}^n$.**

**Proof. Step 1:** Replacing $K$ by a bigger compact, we can assume that $f$ is smoothly extended to a small neighbourhood of the closure $\overline{M\backslash K}$. Then $f$ can be extended to a smooth function on $\mathbb{C}^n$, holomorphic outside of $K$. **Then $\alpha := \overline{\partial}\tilde{f}$ is a $\overline{\partial}$-closed $(0,1)$-form with compact support**.

**Step 2:** Using the standard open embedding of $\mathbb{C}^n$ to $\mathbb{C}P^n$, we may consider $\alpha$ as a $\overline{\partial}$-closed $(0,1)$-form on $\mathbb{C}P^n$. Since $H^1(\mathbb{C}P^n) = 0$, this gives $\alpha = \overline{\partial}\varphi$, where $\varphi$ is a continuous function on $\mathbb{C}P^n$. In particular, **$\varphi$ is bounded on $\mathbb{C}^n \subset \mathbb{C}P^n$.**

**Step 3:** Since $\overline{\partial}\varphi$ vanishes outside of $K$, the function $\varphi$ is holomorphic outside of $K$. Since bounded holomorphic functions on $\mathbb{C}$ are constant, **$\varphi$ is constant on any affine line not intersecting $K$.**

**Step 4:** This implies that $\varphi = \text{const}$ on the union of all affine lines not intersecting $K$. Since $n > 1$, the complement of this set is compact. Substracting constant if necessary, we obtain that **$\varphi$ is a function with compact support**.

**Step 5:** $\overline{\partial}(\tilde{f} - \varphi) = \alpha - \alpha = 0$, **hence $\tilde{f} - \varphi$ is holomorphic**. However, $\varphi$ has compact support, and therefore $f = \tilde{f} - \varphi$ outside of a compact. ■