Hodge theory

lecture 18: Acyclic resolutions

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Sheaves (reminder)

DEFINITION: An open cover of a topological space X is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$.

DEFINITION: A **presheaf** on a topological space M is a collection of vector spaces $\mathcal{F}(U)$, for each open subset $U \subset M$, together with **restriction maps** $R_{UW}\mathcal{F}(U) \longrightarrow \mathcal{F}(W)$ defined for each $W \subset U$, such that for any three open sets $W \subset V \subset U$, $\Psi_{UW} = \Psi_{UV} \circ \Psi_{VW}$. Elements of $\mathcal{F}(U)$ are called **sections of $\mathcal{F}$ over U**, and restriction map often denoted $f|_W$

DEFINITION: A presheaf $\mathcal{F}$ is called **a sheaf** if for any open set U and any cover $U = \bigcup U_I$ the following two conditions are satisfied.

1. Let $f \in \mathcal{F}(U)$ be a section of $\mathcal{F}$ on U such that its restriction to each U_i vanishes. **Then $f = 0$.**

2. Let $f_i \in \mathcal{F}(U_i)$ be a family of sections compatible on the pairwise intersections: $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for every pair of members of the cover. **Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.**

Direct limits

DEFINITION: Commutative diagram of vector spaces is given by the following data. There is a directed graph (graph with arrows). For each vertex of this graph we have a vector space, and each arrow corresponds to a homomorphism of the associated vector spaces. **These homomorphism are compatible, in the following way.** Whenever there exist two ways of going from one vertex to another, the compositions of the corresponding arrows are equal.

DEFINITION: Let $\mathcal{C}$ be a commutative diagram of vector spaces, A, B – vector spaces, corresponding to two vertices of a diagram, and $a \in A, b \in B$ elements of these vector spaces. Write $a \sim b$ if a and b are mapped to the same element $d \in D$ by a composition of arrows from $\mathcal{C}$. Let $\sim$ be an equivalence relation generated by such $a \sim b$. A quotient $\bigoplus_i C_i/E$ is called **a direct limit** of a diagram $\{C_i\}$. The same notion is also called **colimit** and **inductive limit**. Direct limit is denoted $\lim\limits_{\to}$.

DEFINITION: Let $\mathcal{F}$ be a sheaf on M, $x \in M$ a point, and $\{U_i\}$ the set of all neighbourhoods of x. Consider a diagram with the set of vertices indexed by $\{U_i\}$, and arrows from U_i to U_j corresponding to inclusions $U_j \hookrightarrow U_i$. The **space of germs** of $\mathcal{F}$ in x is a direct limit $\lim\limits_{\to} \mathcal{F}(U_i)$ over this diagram. The space of germs is also called **a stalk** of a sheaf.

Sheaf morphisms (reminder)

DEFINITION: Let $\mathcal{B}, \mathcal{B}'$ be sheaves on M. **A sheaf morphism** from $\mathcal{B}$ to $\mathcal{B}'$ is a collection of homomorphisms $\mathcal{B}(U) \longrightarrow \mathcal{B}'(U)$, defined for each open subset $U \subset M$, and compatible with the restriction maps:

$$\begin{array}{ccc} \mathcal{B}(U) & \longrightarrow & \mathcal{B}'(U) \\ \downarrow & & \downarrow \\ \mathcal{B}(U_1) & \longrightarrow & \mathcal{B}'(U_1) \end{array}$$

REMARK: Morphisms of sheaves of modules are defined in the same way, but in this case **the maps $\mathcal{B}(U) \longrightarrow \mathcal{B}'(U)$ should be compatible with the module structure.**

DEFINITION: A sheaf morphism is called **injective** if it is injective on stalks and **surjective**, if it is surjective on stalks.

Čech cohomology

DEFINITION: Let $\mathcal{F}$ be a sheaf on a topological space M and $\{U_i\}$ an open cover of M indexed by a linearly ordered set $\mathcal{I}$. Define the space of **Čech chains**

$$C_{k-1} := \prod_{i_1 < i_2 < ... < i_k} \mathcal{F}\left(\bigcup_{j=1}^{k} U_{i_j}\right).$$

Define **the Čech differential** $d: C_{k-1} \longrightarrow C_k$ mapping $f \in \mathcal{F}\left(\bigcap_{j=1}^{k} U_{i_j}\right)$ to

$$\sum_{i \in \mathcal{I} \backslash \{i_1, ..., i_k\}} (-1)^{\sigma} f\Big|_{U_{i_1} \cap ... \cap U_{i_k} \cap U_i}$$

where $\sigma - 1$ is the number of i in the sequence $i_1 < i_2 < ... < i < ... < i_k$.

Consider the sequence

$$... \xrightarrow{d} C_i \xrightarrow{d} C_{i-1} \xrightarrow{d} ...$$

Its cohomology are called **the Čech cohomology** of the sheaf $\mathcal{F}$, associated with the cover $\{U_i\}$. Elements of $\ker d$ are called **Čech cocycles** and elements of $\operatorname{im} d$ **the Čech coboundaries**.

Čech cohomology and global sections

DEFINITION: A topological space M is called **paracompact** if any open cover of M has a locally finite refinement.

CLAIM: Let A be a sheaf on a paracompact topological space such that its first Čech cohomology vanish for any locally finite covering. Then **for any exact sequence $0 \longrightarrow A \longrightarrow B \xrightarrow{\psi} C \longrightarrow 0$ of sheaves, the sequence $0 \longrightarrow \Gamma(A) \longrightarrow \Gamma(B) \longrightarrow \Gamma(C) \longrightarrow 0$ is exact,** where Γ denotes the space of global sections.

Proof: Let c be a global section of C. Since ψ is surjective, there exists a locally finite (by paracompactness) covering $\{U_i\}$ and $b_i \in B(U_i)$ such that $\psi(b_i) = c\big|_{U_i}$. Then $b_i - b_j\big|_{U_i \cap U_j} \in A(U_i \cap U_j)$ give a Čech 1-cocycle. If it is a coboundary, this means that $b_i - b_j = a_i - a_j$ for some collection of sections $a_i \in A(U_i)$. Then $\tilde{b}_i := b_i - a_i$ agree on pairwise intersections; gluing all $\tilde{b}_i$ to a global section $\tilde{b}$ of B, we obtain that $\psi(b) = c$. ■

Fine sheaves

DEFINITION: Let $\{U_i\}$, be a locally finite open covering of a manifold M, with the closure of U_i compact. Denote by $F^c|_U$ the group of sections of a sheaf F with compact support. **Partition of unity** on a sheaf of rings is a set of sections with compact support $\psi_i \in F^c(U_i)$, such that $\sum_i \psi_i = 1$. A sheaf of rings is called **fine** if it admits a partition of unity for any locally finite covering.

REMARK: The sheaf $C^\infty(M)$ is fine.

CLAIM: Let F be a sheaf of modules over a fine sheaf of rings. Then **the Čech cohomology of F vanish** for any locally finite covering.

Proof: Let $\{U_i\}$ be a covering of M, and $P = \prod_{i_1<i_2<...<i_{k+1}} f_{i_1,...,i_{k+1}} \in F(U_{i_1}\cap ... \cap U_{i_{k+2}})$ a k-cocycle. Consider a partition of unity $\sum \psi_i = 1$ associated with $\{U_i\}$. Then for any i, the product $\psi_i P$ is also a k-cocycle, hence we may assume that P is compactly supported in some U_i, say, U_{i_1}. Put

$$g := \prod_{i_2<...<i_{k+1}} g_{i_2,...,i_{k+1}} \in \prod_{i_2<...<i_{k+1}} F(U_{i_2} \cap ... \cap U_{i_{k+2}})$$

by taking $g_{i_2,...,i_{k+1}} = f_{i_1,i_2,...i_{k+1}}$ and extending $f_{i_1,i_2,...i_{k+1}}$ to $U_{i_2} \cap ... \cap U_{i_{k+2}}$ using compactness of support of $f_{i_1,i_2,...i_{k+1}}$ in U_1. ■

Fine sheaves and flasque sheaves

DEFINITION: Let F be a sheaf such that all restriction maps $F(U) \longrightarrow F(V)$ are surjective. Then F is called **flasque**, or **flabby**.

EXERCISE: Prove that the Čech cohomology of flasque sheaves vanish.

COROLLARY: Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be an exact sequence of sheaves, with A fine or flasque. **Then the sequence of global sections**

$$0 \longrightarrow \Gamma(A) \longrightarrow \Gamma(B) \longrightarrow \Gamma(C) \longrightarrow 0$$

is also exact.

Proof: This follows from vanishing of Čech cohomology, as shown above.

■

Godement resolutions

DEFINITION: Let F be a sheaf on M, and F_x the stalk of F in $x \in M$. It is clearly flasque. Denote by $G(F)$ the sheaf $\prod_{x \in M} F_x$. We consider F as a subsheaf of $G(F)$, and consider the following flasque resolution of $F = F^0$

$$0 \longrightarrow F \xrightarrow{d} F^1 \xrightarrow{d} F^2 \longrightarrow \dots \quad (***)$$

with $F^{i+1} = G(F^i/d(F^{i-1}))$, and d induced by the tautological map

$$F^i \longrightarrow F^i/d(F^{i-1}) \hookrightarrow G(F^i/d(F^{i-1})).$$

The resolution (***) is called **Godement resolution**.

REMARK: The same argument as used for fine sheaves above also proves that **the Čech cohomology of flasqye sheaves vanish.** Therefore, for an exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of sheaves with A flasque, **the sequence of global sections**

$$0 \longrightarrow \Gamma(A) \longrightarrow \Gamma(B) \longrightarrow \Gamma(C) \longrightarrow 0$$

is also exact.

Cohomology of a sheaf

DEFINITION: Let F be a sheaf and $0 \longrightarrow F \xrightarrow{d} F^1 \xrightarrow{d} F^2 \longrightarrow ...$ is Godement resolution. Consider the complex of global sections $0 \longrightarrow \Gamma(F^1) \longrightarrow \Gamma(F^2) \longrightarrow ....$ Its cohomology are called **cohomology of the sheaf F**, denoted $H^i(F)$. **The global sections $\Gamma(F)$ are identified with $H^0(F)$.**

REMARK: Given an exact sequence of sheaves $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, we obtain an exact sequence of their Godement resolutions $0 \longrightarrow A^* \longrightarrow B^* \longrightarrow C^* \longrightarrow 0$, **(prove that it is exact)**. The sequences of sheaves $0 \longrightarrow A^i \longrightarrow B^i \longrightarrow C^i \longrightarrow 0$ gives an exact sequence

$$0 \longrightarrow \Gamma(A^{\geqslant 1}) \longrightarrow \Gamma(B^{\geqslant 1}) \longrightarrow \Gamma(C^{>1}) \longrightarrow 0$$

as shown above. Its cohomology are cohomology of A, B, C. **This gives an exact sequence of cohomology**

$$0 \longrightarrow H^0(A) \longrightarrow H^0(B) \longrightarrow H^0(C) \longrightarrow H^1(A) \longrightarrow H^1(B) \longrightarrow H^1(C) \longrightarrow ...$$

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October 1, 1921 - July 21, 2016

Acyclic resolutions

DEFINITION: A sheaf A on M is called **acyclic** if $H^i(U, A) = 0$ for any opens set $U \subset M$ and any $i > 0$. **An acyclic resolution** for $F = F^0$ is an exact sequence

$$0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow F^2 \longrightarrow ...$$

where all F^i, $i > 0$ are acyclic.

EXAMPLE: Let $x \in M$ and A a vector space. **A skyscraper sheaf** is a sheaf F such that $F(U) = A$ for all $U \ni x$ and $F(U) = 0$ for $U \not\ni x$.

EXERCISE: Prove that **product of skyscraper sheaves is acyclic.** In particular, **the Godement sheaf $G(F)$ is acyclic for any sheaf F.** Prove that **any fine sheaf is also acyclic.**

Further on, we shall prove the following resultat.

THEOREM: Let $0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow F^2 \longrightarrow ...$ be an acyclic resolution for a sheaf $F = F^0$. **Then cohomology of the complex $0 \longrightarrow \Gamma(F^1) \longrightarrow \Gamma(F^2) \longrightarrow ...$ are equal to $H^*(F)$.**

Morphisms of complexes

DEFINITION: A complex is a sequence of objects of abelian category (sheaves, groups, modules) ... $\xrightarrow{d} C^{i-1} \xrightarrow{d} C^i \xrightarrow{d} C^{i+1} \xrightarrow{d}$ ..., $i \in \mathbb{Z}$, with $d^2 = 0$. **Cohomology** of a complex is $\ker d / \operatorname{im} d$. **A morphism** of complexes $(C^i, d) \longrightarrow (C_1^i, d)$ is a sequence of maps $\psi_i : C^i \longrightarrow C_1^i$ commuting with d. **Category of complexes is also abelian.**

EXERCISE: Let $0 \longrightarrow A^* \longrightarrow B^* \longrightarrow C^* \longrightarrow 0$ be an exact sequence of complexes. **Prove that there exists a long exact sequence**

$$... \longrightarrow H^i(A) \longrightarrow H^i(B) \longrightarrow H^i(C) \longrightarrow H^{i+1}(A) \longrightarrow H^{i+1}(B) \longrightarrow H^{i+1}(C) \longrightarrow ...$$

DEFINITION: A morphism of complexes is called **quasi-isomorphism** if it induces an isomorphism on cohomology.

Cones of morphisms

DEFINITION: Let $(F^i, d_F) \xrightarrow{\psi_i} (G^i, d_G)$ be a morphism of complexes. **The cone** $C(\psi)$ is a complex $F^{i+1} \oplus G^i$, with differential given by $d_F + d_G + (-1)^i \psi_{i+1}$.

REMARK: Denote by $F^*[1]$ the complex (F^{i+1}, d), that is, F^* shifted by 1. Since the sequence of complexes $0 \longrightarrow G^* \longrightarrow C(\psi) \longrightarrow F^*[1] \longrightarrow 0$ is exact, **we obtain an exact sequence**

$$... \longrightarrow H^i(G) \longrightarrow H^i(C(\psi)) \longrightarrow H^{i+1}(F) \longrightarrow H^{i+1}(G) \longrightarrow H^{i+1}(C(\psi)) \longrightarrow ...$$

COROLLARY: A morphism of complexes is a quasi-isomorphism if and only if its cone has zero cohomology.

Exercise 1: Let $0 \longrightarrow C^1 \longrightarrow C^2 \longrightarrow ...$ be an exact sequence of acyclic sheaves. **Prove that the sequence of global sections $0 \longrightarrow \Gamma(C^1) \longrightarrow \Gamma(C^2) \longrightarrow ...$ is also exact.**

Cones and cohomology

CLAIM: Let $0 \longrightarrow A_1^0 \longrightarrow A_1^1 \longrightarrow A_1^2 \longrightarrow ...$ be an acyclic resolution for a sheaf A, and $0 \longrightarrow A_2^0 \longrightarrow A_2^1 \longrightarrow A_2^2 \longrightarrow ...$ another acyclic resolution. Suppose that there exists a morphism φ of complexes inducing identity on $A_2^0 = A_1^0 = A$. **Then the cohomology of the complex $\Gamma(A_1^*)$ are equal to the cohomology of $\Gamma(A_2^*)$.**

Proof. Step 1: Consider the complex X^*, given by $0 \longrightarrow A_1^1 \longrightarrow A_1^2 \longrightarrow ...$ and Y^*, given by $0 \longrightarrow A_2^1 \longrightarrow A_2^2 \longrightarrow ...$ (we drop the first term $A_2^0 = A_1^0 = A$). Then the cohomology sheaves $\mathcal{H}^i(\cdot)$ of these complexes are equal to A in 0, and vanish in other terms. The map φ induces a morphism of complexes $X^* \xrightarrow{\varphi} Y^*$ which induces identity on the cohomology sheaves $H^0(A_2^*) = H^0(A_1^*) = A$. The long exact sequence
$... \longrightarrow \mathcal{H}^i(A_1^*) \longrightarrow \mathcal{H}^i(A_2^*) \longrightarrow \mathcal{H}^i(C(\varphi)) \longrightarrow ...$ implies that **the cone $C(\varphi)$ is an exact complex of acyclic sheaves.**

Step 2: Exercise 1 implies that the sequence
$... \xrightarrow{d} \Gamma(C^i(\varphi)) \xrightarrow{d} \Gamma(C^{i+1}(\varphi)) \xrightarrow{d} ...$ is exact. However, this sequence is a complex of vector spaces, obtained as a cone of a morphism of complexes $\Gamma(A_1^*) \longrightarrow \Gamma(A_2^*)$, and **from the cone exact sequence we obtain that cohomology of these complexes are equal.** ■

Bicomplexes

DEFINITION: Bicomplex is a collection $C^{i,j}$ of objects in abelian category, enumerated by $i,j \in \mathbb{Z}^2$, and equipped with two differentials $d^{1,0}: C^{i,j} \longrightarrow C^{i+1,j}$ and $d^{0,1}: C^{i,j} \longrightarrow C^{i,j+1}$, anti-commuting and satisfying $(d^{0,1})^2 = 0$ and $(d^{1,0})^2 = 0$.

DEFINITION: Totalization of a bicomplex $(C^{i,j}, d^{1,0}, d^{0,1})$ is a complex $\operatorname{Tot}^*(C^{i,j}, d)$ with $d = d^{1,0} + d^{0,1}$ and $\operatorname{Tot}^p(C^{i,j}) = \bigoplus_{i+j=p} C^{i,j}$.

Exercise 2: Let $(C^{i,j}, d^{1,0}, d^{0,1})$ be a bicomplex, with $i,j \geqslant 0$. Suppose that cohomology of $d^{1,0}$ are equal 0. **Prove that cohomology of $\operatorname{Tot}^*(C^{i,j})$ vanish.**

Bicomplexes (2)

Exercise 2: Let $(C^{i,j}, d^{1,0}, d^{0,1})$ be a bicomplex, with $i,j \geqslant 0$. Suppose that cohomology of $d^{1,0}$ are equal 0. **Prove that cohomology of $\operatorname{Tot}^*(C^{i,j})$ vanish.**

Claim 1: Let $(C^{i,j}, d^{1,0}, d^{0,1})$ be a bicomplex, with $i,j \geqslant 0$. Suppose that cohomology of $(C^{i,*}, d^{0,1})$ vanish for all $i > 0$. **Then the cohomology of $\operatorname{Tot}^*(C^{i,j})$ are equal to cohomology of $(C^{0,*}, d^{0,1})$.**

Proof: Consider the natural surjective morphism of complexes $\operatorname{Tot}^*(C^{i,j}) \xrightarrow{\Psi} (C^{0,*}, d^{0,1})$. Then $\ker\Psi = \operatorname{Tot}^*_{i>0}(C^{i,j})$, where $\operatorname{Tot}^*_{i>0}(C^{i,j})$ is totalization of the subcomplex $(C^{*+1,*}, d^{1,0}, d^{0,1}) \subset (C^{*,*}, d^{1,0}, d^{0,1})$. By Exercise 2, cohomology of $\operatorname{Tot}^*_{i>0}(C^{i,j})$ vanish. Taking the long exact sequence associated with the exact sequence of complexes

$$0 \longrightarrow \operatorname{Tot}^*_{i>0}(C^{i,j}) \longrightarrow \operatorname{Tot}^*(C^{i,j}) \longrightarrow C^{0,*} \longrightarrow 0$$

we obtain that cohomology of $(C^{0,*}, d^{0,1})$ are equal to the cohomology of $(\operatorname{Tot}^*(C^{i,j}), d^{1,0} + d^{0,1})$. ■

Godement bicomplex

Let $A = A^0$ be a sheaf and $0 \longrightarrow A^0 \longrightarrow A^1 \longrightarrow A^2 \longrightarrow ...$ an acyclic resolution, and $G^n(A^i)$ the n-th term of Godement resolution for A^i. This gives a bicomplex $G^{*,*}$

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & A^2 & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G^1(A^0) & \longrightarrow & G^1(A^1) & \longrightarrow & G^1(A^2) & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G^2(A^0) & \longrightarrow & G^2(A^1) & \longrightarrow & G^2(A^2) & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}
$$

with all sheaves acyclic except A^0.

Acyclic sheaves

DEFINITION: A sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of sheaves is called **an exact sequence** if the corresponding sequences of stalks are exact.

DEFINITION: A functor Φ from sheaves to vector spaces is called **left exact** if any exact sequence of sheaves $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is mapped to a left exact sequence $0 \longrightarrow \Phi(A) \longrightarrow \Phi(B) \longrightarrow \Phi(C)$.

EXAMPLE: Functor of global sections $\mathcal{F} \longrightarrow \Gamma_M(\mathcal{F})$ is left exact.

DEFINITION: A sheaf is called **acyclic** if for any open set $U \subset M$ and any exact sequence of sheaves $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, the sequence

$$0 \longrightarrow \Gamma_U(A) \longrightarrow \Gamma_U(B) \longrightarrow \Gamma_U(C) \longrightarrow 0$$

is exact.

REMARK: As shown above, **a sheaf A is acyclic if its Čech cohomology $H^1(A)$ vanish** for any locally finite covering. In particular, all sheaves of modules over $C^\infty M$ are acyclic.

DEFINITION: Let $0 \longrightarrow F \longrightarrow F_1 \longrightarrow F_2 \longrightarrow ...$ be an exact sequence of sheaves. Assume that all F_i are acyclic. Then this sequence is called **an acyclic resolution** for F.

Sheaf cohomology

DEFINITION: Let $0 \longrightarrow F \longrightarrow F_1 \longrightarrow F_2 \longrightarrow ...$ be an acyclic resolution of F. **Cohomology group** $H^i(F)$ is defined as i-th group of cohomology of the corresponding complex of global sections

$$\Gamma_M(F) \longrightarrow \Gamma_M(F_1) \longrightarrow \Gamma_M(F_2) \longrightarrow ...$$

PROPOSITION: (Properties of cohomology sheaves):
1. The groups $H^i(F)$ **don't depend on the choice of acyclic resolution**.
2. $H^i(F) = 0$ **for all** $i > 0$ **if and only if** F **is acyclic.**
3. For any exact sequence of sheaves $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ there is **a long exact sequence**

$$0 \longrightarrow \Gamma(A) \longrightarrow \Gamma(B) \longrightarrow \Gamma(C) \longrightarrow H^1(A) \longrightarrow H^1(B) \longrightarrow H^1(C) \longrightarrow ...$$

Proof is later today.

Independence of cohomology

REMARK: The following theorem implies that the **cohomology of a sheaf are independent from the choice of acyclic resolution.**

THEOREM: Let $A = A^0$ and $0 \longrightarrow A^0 \longrightarrow A^1 \longrightarrow ...$ be an acyclic resolution of A. Then **the cohomology of $\Gamma(A^i)$ are equal to the cohomology of the global sections of the Godement resolution $\Gamma(G^i(A^0))$.**

Proof: Apply the functor $\Gamma(\cdot)$ to the bicomplex $G^{*,*}$. Exercise 1 implies that the columns and rows of $\Gamma(G^{*,*})$ are exact, with the possible exception of $\Gamma(G^{0,*})$ and $\Gamma(G^{*,0})$. Then Claim 1 implies that cohomology of totalization of $\Gamma(G^{*,*})$ are equal to the cohomology of $\Gamma(G^{0,*}), d^{0,1}$, which is cohomology of $\Gamma(G^*(A^0))$ and to the cohomology of $\Gamma(G^{*,0}), d^{1,0}$ which is the same as cohomology of $\Gamma(A^*)$. ■

Dolbeault resolution

REMARK: From Poincaré-Dolbeault-Grothendieck lemma we obtain **an acyclic resolution**

$$0 \longrightarrow \Omega^p(M) \hookrightarrow \Lambda^{p,0}(M) \xrightarrow{\overline{\partial}} \Lambda^{p,1}(M) \xrightarrow{\overline{\partial}} \Lambda^{p,2}(M) \xrightarrow{\overline{\partial}} ... \quad (****)$$

of the sheaf of holomorphic p-forms. Indeed, the kernel of $\Lambda^{p,0}(M) \xrightarrow{\overline{\partial}} \Lambda^{p,1}(M)$ is forms with holomorphic coefficients; other terms of (**) are exact by Poincaré-Dolbeault-Grothendieck lemma. The sheaves $\Lambda^{p,0}(M)$ are all sheaves of $C^\infty(M)$-modules, hence they are acyclic.

COROLLARY: Let M be a compact Kähler manifold. Then **the space $H^{p,q}(M)$ is identified with the cohomology group $H^q(\Omega^p(M))$.**

Proof: Indeed, the cohomology of $\Gamma(\cdot)$ applied to (****) is the kernel of the corresponding Laplacian $\Delta_{\overline{\partial}}$, which is the same as the kernel of Δ_d on $H^{p,*}(M)$. ■