Hodge theory

lecture 19: Chern connections

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Curvature

DEFINITION: Let $\nabla: B \longrightarrow B \otimes \Lambda^1 M$ be a connection on a vector bundle B. We extend ∇ to an operator

$$V \xrightarrow{\nabla} \Lambda^1(M) \otimes V \xrightarrow{\nabla} \Lambda^2(M) \otimes V \xrightarrow{\nabla} \Lambda^3(M) \otimes V \xrightarrow{\nabla} ...$$

using the Leibnitz identity $\nabla(\eta \otimes b) = d\eta + (-1)^{\tilde{\eta}} \eta \wedge \nabla b$. Then the operator $\nabla^2: B \longrightarrow B \otimes \Lambda^2(M)$ is called **the curvature** of ∇.

REMARK: The algebra of differential forms with coefficients in $\operatorname{End} B$ **acts on** $\Lambda^* M \otimes B$ via $\eta \otimes a(\eta' \otimes b) = \eta \wedge \eta' \otimes a(b)$, where $a \in \operatorname{End}(B)$, $\eta, \eta' \in \Lambda^* M$, and $b \in B$.

REMARK: $\nabla^2(fb) = d^2 fb + df \wedge \nabla b - df \wedge \nabla b + f \nabla^2 b$, hence **the curvature is a** $C^\infty M$**-linear operator. We shall consider the curvature** B **as a 2-form with values in** $\operatorname{End} B$. Then $\nabla^2 := \Theta_B \in \Lambda^2 M \otimes \operatorname{End} B$, where an $\operatorname{End}(B)$-valued form acts on $\Lambda^* M \otimes B$ as above.

Flat bundles

DEFINITION: Let (B, ∇) be a bundle with connection. **Holonomy group** of γ is the group of endomorphisms of the fiber B_x obtained from parallel transports along all paths starting and ending in $x \in M$

DEFINITION: A bundle is **flat** if and only if its curvature vanishes.

The following theorem easily follows from the Frobenius theorem and interpretation of connection on B as of splitting of the tangent bundle of the total space of the principal $GL(n)$-fibration associated with B.

THEOREM: Let (B, ∇) be a vector bundle with connection over a simply connected manifold. **Then B is flat if and only if its holonomy group is trivial**.

Holomorphic bundles

DEFINITION: Holomorphic vector bundle on a complex manifold M is a locally trivial sheaf of $\mathcal{O}_M$-modules.

DEFINITION: The total space $\operatorname{Tot}(B)$ of a holomorphic bundle B over M is the space of all pairs $\{x \in M, b \in B_x/\mathfrak{m}_x B\}$, where B_x is the stalk of B in $x \in M$ and $\mathfrak{m}_x$ the maximal ideal of x. We equip $\operatorname{Tot}(B)$ with the natural topology and holomorphic structure, in such a way that $\operatorname{Tot}(B)$ becomes a locally trivial holomorphic fibration with fiber $\mathbb{C}^r$, $r = \operatorname{rk} B$.

REMARK: The set of holomorphic sections of a map $\operatorname{Tot}(B) \longrightarrow M$ is naturally identified with the set of sections of the sheaf B.

CLAIM: Ler B be a holomorphic bundle. Consider the sheaf $B_{C^\infty} := B \otimes_{\mathcal{O}_M} C^\infty M$. **Then B_{C^∞} is a locally trivial sheaf of $C^\infty M$-modules**.

DEFINITION: B_{C^∞} is called **smooth vector bundle underlying the holomorphic vector bundle B**.

REMARK: The natural map $\operatorname{Tot}(B) \longrightarrow \operatorname{Tot}(B_{C^\infty})$ is a diffeomorphism.

$\overline{\partial}$-operator on vector bundles

REMARK: Let M be a complex manifold. Then **the operator $\overline{\partial}: C^\infty M \longrightarrow \Lambda^{0,1}(M)$ is $\mathcal{O}_M$-linear.**

DEFINITION: Let B be a holomorphic vector bundle on M. Consider an operator $\overline{\partial}: B_{C^\infty} \longrightarrow B_{C^\infty} \otimes \Lambda^{0,1}(M)$ mapping $b \otimes f$ to $b \otimes \overline{\partial} f$, where b is a holomorphic section of B, and f smooth. This operator is called **a holomorphic structure operator** on B. **It is well-defined because $\overline{\partial}$ is $\mathcal{O}_M$-linear**, and $B_{C^\infty} = B \otimes_{\mathcal{O}_M} C^\infty M$.

REMARK: The kernel of $\overline{\partial}: B_{C^\infty} \longrightarrow B_{C^\infty} \otimes \Lambda^{0,1}(M)$ **coincides with the image of B** under the natural sheaf embedding $B \hookrightarrow B_{C^\infty}$, with $b \longrightarrow b \otimes 1$.

DEFINITION: A **$\overline{\partial}$-operator** on a smooth complex vector bundle V over a complex manifold is a differential operator $V \xrightarrow{\overline{\partial}} \Lambda^{0,1}(M) \otimes V$ satisfying $\overline{\partial}(fb) = \overline{\partial}(f) \otimes b + f\overline{\partial}(b)$ for any $f \in C^\infty M, b \in V$.

REMARK: A $\overline{\partial}$-operator **can be extended to**

$$\overline{\partial}: \Lambda^{0,i}(M) \otimes V \longrightarrow \Lambda^{0,i+1}(M) \otimes V,$$

using the Leibnitz identity $\overline{\partial}(\eta \otimes b) = \overline{\partial}(\eta) \otimes b + (-1)^{\tilde{\eta}} \eta \wedge \overline{\partial}(b)$, for all $b \in V$ and $\eta \in \Lambda^{0,i}(M)$.

Complexification (reminder)

DEFINITION: Let $M_{\mathbb R}$ be a real analytic manifold, and $M_{\mathbb C}$ a complex analytic manifold equipped with an antiholomorphic involution, such that $M_{\mathbb R}$ is the set of its fixed points. Then $M_{\mathbb C}$ is called **complexification** of $M_{\mathbb R}$.

DEFINITION: A tensor on a real analytic manifold is called **real analytic** if it is expressed locally by a sum (infinite, generally speaking) of coordinate monomials with real analytic coefficients.

CLAIM: Let $M_{\mathbb R}$ be a real analytic manifold, $M_{\mathbb C}$ its complexification, and Φ a tensor on $M_{\mathbb R}$. **Then Φ is real analytic if and only if Φ can be extended to a holomorpic tensor $\Phi_{\mathbb C}$ in some neighbourhood of $M_{\mathbb R}$ inside $M_{\mathbb C}$.**

Proof: The "if" part is clear, because every complex analytic tensor on $M_{\mathbb C}$ is by definition real analytic on $M_{\mathbb R}$.

Conversely, suppose that Φ is expressed by a sum of coordinate monomials with real analytic coefficients f_i. Let $\{U_i\}$ be a cover of M, and $\tilde U_i := U_i \times B_\varepsilon$ the corresponding cover of a neighbourhood of $M_{\mathbb R}$ in $M_{\mathbb C}$ constructed above. Chosing ε sufficiently small, we can assume that the Taylor series giving coefficients of Φ converges on each $\tilde U_i$. **We define $\Phi_{\mathbb C}$ as the sum of these series.** ■

Holomorphic structure operator

REMARK: For any holomorphic bundle, one has $\overline{\partial}^2 = 0$. Indeed, a holomorphic bundle admits a local trivialization.

THEOREM: (Malgrange) Let $\overline{\partial}: V \longrightarrow \Lambda^{0,1}(M) \otimes V$ be a $\overline{\partial}$-operator on a complex vector bundle, satisfying $\overline{\partial}^2 = 0$, where $\overline{\partial}$ is extended to

$$V \xrightarrow{\overline{\partial}} \Lambda^{0,1}(M) \otimes V \xrightarrow{\overline{\partial}} \Lambda^{0,2}(M) \otimes V \xrightarrow{\overline{\partial}} \Lambda^{0,3}(M) \otimes V \xrightarrow{\overline{\partial}} ...$$

as above. **Then $B := \ker \overline{\partial} \subset V$ is a holomorphic bundle of the same rank, and $V = B_{\mathbb{C}^\infty}$.**

Proof: We prove this theorem in additional assumption that V and $\overline{\partial}$ is real analytic. This assumption can be justified using the Newlander-Nirenberg theorem.

Let $M_{\mathbb{C}} \subset M \times \overline{M}$ be a small neighbourhood of diagonal, considered as a complexification of M, and $V_{\mathbb{C}}$ a holomorphic vector bundle on $M_{\mathbb{C}}$ obtained from the real vector bundle V. We extend $\overline{\partial}$ to a differential operator $\overline{\partial}_{\mathbb{C}}$ on $V_{\mathbb{C}}$. Then $\overline{\partial}_{\mathbb{C}}$ is a connection in the restriction of $V_{\mathbb{C}}$ to the fibers of the natural projection $\pi: M_{\mathbb{C}} \longrightarrow M$, and the condition $\overline{\partial}^2 = 0$ implies that it is flat. **The sheaf $\ker \overline{\partial}$ constant on fibers of π,** hence $B = \pi_*(\ker \overline{\partial})$ is a holomorphic bundle of the same rank as V. ■

Connections and holomorphic structures

DEFINITION: Let V be a smooth complex vector bundle with connection $\nabla : V \longrightarrow \Lambda^1(M) \otimes V$ and holomorphic structure $\overline{\partial} : V \longrightarrow \Lambda^{0,1}(M) \otimes V$. Consider the Hodge type decomposition of ∇, $\nabla = \nabla^{0,1} + \nabla^{1,0}$, where

$$\nabla^{0,1} : V \longrightarrow \Lambda^{0,1}(M) \otimes V, \quad \nabla^{1,0} : V \longrightarrow \Lambda^{1,0}(M) \otimes V.$$

We say that **the connection ∇ is compatible with the holomorphic structure** if $\nabla^{0,1} = \overline{\partial}$.

DEFINITION: A holomorphic Hermitian vector buncle is a smooth complex vector bundle equipped with a Hermitian metric and a holomorphic structure.

DEFINITION: Chern connection on a holomorphic Hermitian vector bundle is a unitary connection compatible with the holomorphic structure.

Chern connection

THEOREM: Every holomorphic Hermitian vector bundle **admits a Chern connection, which is unique.**

Proof. Step 1: Given a complex vector bundle B, define **complex conjugate bundle** $\overline{B}$ as the same $\mathbb{R}$-bundle with complex conjugate $\mathbb{C}$-action. Then **a connection ∇ on B defines a connection $\overline{\nabla}$ on $\overline{B}$, with $\overline{\nabla}^{1,0} = \overline{\nabla^{0,1}}$ and $\overline{\nabla}^{0,1} = \overline{\nabla^{1,0}}$.**

Step 2: Define **$\nabla^{1,0}$-operator** on a complex vector bundle B as a map $B \xrightarrow{\nabla^{1,0}} \Lambda^{1,0}(M) \otimes B$, satisfying $\Lambda^{1,0}(fb) = \partial(f) \otimes b + f\nabla^{1,0}(b)$ for any $f \in C^\infty M, b \in B$. **A $\overline{\partial}$-operator on B defines an $\nabla^{1,0}$-operator on $\overline{B}$, and vice versa**.

Step 3: Hermitian form defines an isomorphism of complex vector bundles $B \xrightarrow{g} \overline{B}^*$. Holomorphic structure on B defines a $\overline{\partial}$-operator on $\overline{B} = B^*$, which is the same as $\nabla^{1,0}$-operator $\nabla_g^{1,0}$ on B. **This gives a connection operator $\nabla := \overline{\partial} + \nabla_g^{1,0}$ on B,** which is Hermitian by construction. ■