Hodge theory

Lecture 20: Curvature of line bundles

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Curvature (reminder)

DEFINITION: Let $\nabla: B \longrightarrow B \otimes \Lambda^1 M$ be a connection on a vector bundle B. We extend ∇ to an operator

$$V \xrightarrow{\nabla} \Lambda^1(M) \otimes V \xrightarrow{\nabla} \Lambda^2(M) \otimes V \xrightarrow{\nabla} \Lambda^3(M) \otimes V \xrightarrow{\nabla} ...$$

using the Leibnitz identity $\nabla(\eta \otimes b) = d\eta + (-1)^{\tilde\eta}\eta \wedge \nabla b$. Then the operator $\nabla^2: B \longrightarrow B \otimes \Lambda^2(M)$ is called **the curvature** of ∇.

REMARK: **The algebra of differential forms with coefficients in** $\operatorname{End} B$ **acts on** $\Lambda^* M \otimes B$ via $\eta \otimes a(\eta' \otimes b) = \eta \wedge \eta' \otimes a(b)$, where $a \in \operatorname{End}(B)$, $\eta, \eta' \in \Lambda^* M$, and $b \in B$.

REMARK: $\nabla^2(fb) = d^2 fb + df \wedge \nabla b - df \wedge \nabla b + f\nabla^2 b$, hence **the curvature is a** $C^\infty M$**-linear operator.** **We shall consider the curvature** B **as a 2-form with values in** $\operatorname{End} B$. Then $\nabla^2 := \Theta_B \in \Lambda^2 M \otimes \operatorname{End} B$, where an $\operatorname{End}(B)$-valued form acts on $\Lambda^* M \otimes B$ as above.

Holomorphic bundles (reminder)

DEFINITION: **Holomorphic vector bundle** on a complex manifold M is a locally trivial sheaf of $\mathcal{O}_M$-modules.

DEFINITION: **The total space** $\operatorname{Tot}(B)$ of a holomorphic bundle B over M is the space of all pairs $\{x \in M, b \in B_x/\mathfrak{m}_x B\}$, where B_x is the stalk of B in $x \in M$ and $\mathfrak{m}_x$ the maximal ideal of x. We equip $\operatorname{Tot}(B)$ with the natural topology and holomorphic structure, in such a way that $\operatorname{Tot}(B)$ becomes a locally trivial holomorphic fibration with fiber $\mathbb{C}^r$, $r = \operatorname{rk} B$.

REMARK: **The set of holomorphic sections of a map $\operatorname{Tot}(B) \longrightarrow M$ is naturally identified with the set of sections of the sheaf B.**

CLAIM: Ler B be a holomorphic bundle. Consider the sheaf $B_{C^\infty} := B \otimes_{\mathcal{O}_M} C^\infty M$. **Then B_{C^∞} is a locally trivial sheaf of $C^\infty M$-modules**.

DEFINITION: B_{C^∞} is called **smooth vector bundle underlying the holomorphic vector bundle B**.

REMARK: The natural map $\operatorname{Tot}(B) \longrightarrow \operatorname{Tot}(B_{C^\infty})$ is a diffeomorphism.

$\overline{\partial}$-operator on vector bundles (reminder)

DEFINITION: Let B be a holomorphic vector bundle on M. Consider an operator $\overline{\partial}: B_{C^\infty} \longrightarrow B_{C^\infty} \otimes \Lambda^{0,1}(M)$ mapping $b \otimes f$ to $b \otimes \overline{\partial} f$, where b is a holomorphic section of B, and f smooth. This operator is called **a holomorphic structure operator** on B. **It is well-defined because $\overline{\partial}$ is $\mathcal{O}_M$-linear**, and $B_{C^\infty} = B \otimes_{\mathcal{O}_M} C^\infty M$.

DEFINITION: A **$\overline{\partial}$-operator** on a smooth complex vector bundle V over a complex manifold is a differential operator $V \xrightarrow{\overline{\partial}} \Lambda^{0,1}(M) \otimes V$ satisfying $\overline{\partial}(fb) = \overline{\partial}(f) \otimes b + f\overline{\partial}(b)$ for any $f \in C^\infty M, b \in V$.

REMARK: A $\overline{\partial}$-operator **can be extended to $\overline{\partial}: \Lambda^{0,i}(M) \otimes V \longrightarrow \Lambda^{0,i+1}(M) \otimes V$**, using the Leibnitz identity $\overline{\partial}(\eta \otimes b) = \overline{\partial}(\eta) \otimes b + (-1)^{\tilde{\eta}} \eta \wedge \overline{\partial}(b)$, for all $b \in V$ and $\eta \in \Lambda^{0,i}(M)$.

THEOREM: (Malgrange) Let $\overline{\partial}: V \longrightarrow \Lambda^{0,1}(M) \otimes V$ be a $\overline{\partial}$-operator on a complex vector bundle, satisfying $\overline{\partial}^2 = 0$, where $\overline{\partial}$ is extended to

$$V \xrightarrow{\overline{\partial}} \Lambda^{0,1}(M) \otimes V \xrightarrow{\overline{\partial}} \Lambda^{0,2}(M) \otimes V \xrightarrow{\overline{\partial}} \Lambda^{0,3}(M) \otimes V \xrightarrow{\overline{\partial}} \ldots$$

as above. **Then $B := \ker \overline{\partial} \subset V$ is a holomorphic bundle of the same rank, and $V = B_{\mathbb{C}^\infty}$.**

Chern connection (reminder)

DEFINITION: Let V be a smooth complex vector bundle with connection $\nabla : V \longrightarrow \Lambda^1(M) \otimes V$ and holomorphic structure $\overline{\partial} : V \longrightarrow \Lambda^{0,1}(M) \otimes V$. Consider the Hodge type decomposition of ∇, $\nabla = \nabla^{0,1} + \nabla^{1,0}$, where

$$\nabla^{0,1} : V \longrightarrow \Lambda^{0,1}(M) \otimes V, \quad \nabla^{1,0} : V \longrightarrow \Lambda^{1,0}(M) \otimes V.$$

We say that **the connection ∇ is compatible with the holomorphic structure** if $\nabla^{0,1} = \overline{\partial}$.

DEFINITION: A holomorphic Hermitian vector buncle is a smooth complex vector bundle equipped with a Hermitian metric and a holomorphic structure.

DEFINITION: Chern connection on a holomorphic Hermitian vector bundle is a unitary connection compatible with the holomorphic structure.

THEOREM: Every holomorphic Hermitian vector bundle **admits a Chern connection, which is unique.**

REMARK: When people say about "curvature of a holomorphic Hermitian line bundle", **they speak about curvature of a Hermitian connection.**

Bianchi identity

REMARK: $[\nabla, \{\nabla, \nabla\}] = [\{\nabla, \nabla\}, \nabla] + [\nabla, \{\nabla, \nabla\}] = 0$ by the super Jacobi identity. This gives **the Bianchi identity:** $\nabla(\Theta_B \wedge \eta) = \Theta_B \wedge \nabla(\eta)$.

REMARK: When B is a line bundle, $\operatorname{End}(B)$ is trivial, and Θ_B is a 2-form.

CLAIM: A curvature of a line bundle is a closed 2-form.

Proof: For any form $\theta \in \Lambda^i(M) \otimes \operatorname{End}(B)$, Leibnitz identity gives $\nabla(\theta \wedge \eta) = d\theta \wedge \eta + (-1)^i \theta \wedge \nabla(\eta)$. Bianchi identity gives $\nabla(\Theta_B \wedge \eta) = \Theta_B \wedge \nabla(\eta)$. Therefore $d\Theta_B = 0$. ■

REMARK: The same argument can be used to show that **$\operatorname{Tr}_B \Theta_B^i$ is a closed $2i$-form**, where Tr_B denotes the trace in $\operatorname{End}(B)$, and Θ_B^i is the i-th power of an $\operatorname{End}(B)$-valued form.

DEFINITION: Cohomology classes of $\operatorname{Tr}_B \Theta_B^i$ are called **characteristic vlasses** of a bundle B ("Chern-Weil formula"). When B is a line bundle, the cohomology class of $-\frac{\sqrt{-1}}{\pi}\Theta_B$ is called **first Chern class of B**, denoted $c_1(B)$.

Luigi Bianchi

Luigi Bianchi (1856 - 1928)

Exponential sequence

REMARK: Let B be a line bundle on a manifold, $\{U_\alpha\}$ its cover where B is trivialized, and $\varphi_{\alpha\beta}$ the corresponding transition functions defined on $U_\alpha \cap U_\beta$. On each intersection $U_\alpha \cap U_\beta \cap U_\gamma$ we have $\varphi_{\alpha\beta}\varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$, hence **a trivialization of B on $\{U_\alpha\}$ defines a Čech 1-cocycle on B with values in $(C^\infty M)^*$.**

The following claim is clear from the definitions.

CLAIM: Isomorphism classes of vector bundles are in bijective correspondence with $H^1(M,(C^\infty M)^*)$.

DEFINITION: Exponential exact sequence is the following exact sequence of sheaves:

$$0 \longrightarrow \mathbb{Z}_M \longrightarrow C^\infty M \longrightarrow (C^\infty M)^* \longrightarrow 0,$$

Since $H^i(C^\infty M) = 0$ for $i > 0$, **the corresponding long exact sequence gives $0 \longrightarrow H^1(M,(C^\infty M)^*) \xrightarrow{\sim} H^2(M,\mathbb{Z}) \longrightarrow 0$.**

Line bundles on $\mathbb{C}P^\infty$

EXERCISE: Let B be a vector bundle on M. **Prove that $B \oplus B'$ is trivial for some bundle B'.**

CLAIM: For any complex line bundle L on M, **there exists a map $\varphi : M \longrightarrow \mathbb{C}P^n$ such that $L = \pi^* \mathcal{O}(-1)$.**

Proof: Indeed, suppose that $L \oplus B'$ is trivial, $B_1 = L \oplus B' = V \otimes_{\mathbb{C}} C^\infty M$. Then each point $x \in M$ defines a line $\varphi(x) \in \mathbb{P}V$ such that $L|_x = \varphi(x) \subset B_1|_x = V$. In this situation, **$L$ is obtained as a pullback of the tautological vector bundle.** ■

First Chern class

DEFINITION: The isomorphism $H^1(M,(C^\infty M)^*) \xrightarrow{\sim} H^2(M,\mathbb{Z})$ maps a line bundle $L \in H^1(M,(C^\infty M)^*)$ to its **integer Chern class** $c_1^{\mathbb{Z}}(B) \in H^2(M,\mathbb{Z})$

THEOREM: (Gauss-Bonnet)
Let L be a line bundle on M. Then the natural map $H^2(M,\mathbb{Z}) \longrightarrow H^2(M,\mathbb{R})$ **maps $c_1^{\mathbb{Z}}(L)$ to the class $c_1(L) = -\frac{\sqrt{-1}}{\pi}[\Theta_L]$ defined above.**

Proof: Let $\mathcal{O}(-1)$ be the tautological bundle on $\mathbb{C}P^n$. As shown above, any line bundle L on M can be obtained as $\varphi^*(\mathcal{O}(-1))$, hence it suffices to prove $[\Theta_L] = \sqrt{-1}\,\pi c_1^{\mathbb{Z}}(L)$ for $L = \mathcal{O}(-1)$ on $\mathbb{C}P^n$. Since a cohomology class in $H^2(\mathbb{C}P^n)$ is determined by its restriction to $\mathbb{C}P^1$, it would suffice to prove this formula for $L = \mathcal{O}(-1)$ on $\mathbb{C}P^1$. In this case, $c_1^{\mathbb{Z}}(L)$ is the Euler characteristic of L, and $[\Theta_L] = \sqrt{-1}\pi c_1^{\mathbb{Z}}(L)$ is the usual Gauss-Bonnet formula on a 2-sphere, which can be obtained by computing the volume of S^2. ■

Real structures on a complex vector space

DEFINITION: Real structure on a complex vector space is anticomplex involution.

EXERCISE: For any complex vector space V and a real structure ι, denote by $V_{\mathbb R}$ the fixed point set of ι. **Prove that $V = V_{\mathbb R} \otimes_{\mathbb R} \mathbb C$.**

EXAMPLE: Let V be a Hermitian vector space, andEnd$_{\mathbb C} V$ its endomorphism space. Consider the real structure $\varphi \xrightarrow{\iota} -\varphi^*$, where φ^* denotes the Hermitian conjugate. **Prove that the fixed point set of ι is the space of anti-Hermitian matrices $\mathfrak u(V)$.**

Curvature of the Chern connection

PROPOSITION: Curvaure Θ_B of a Chern connection on B is a (1,1)-form: $\Theta_B \in \Lambda^{1,1}(M) \otimes \operatorname{End}(B)$.

Proof. Step 1: Let B be a Hermitan bundle. Consider the operator $\varphi \xrightarrow{\iota} -\varphi^*$ acting on $\operatorname{End}(B)$, where $\varphi \longrightarrow \varphi^*$ denotes the Hermitian conjugation. Since $\iota^2 = \operatorname{Id}$, and this is an anticomplex operator, it defines the real structure, and its fixed point set is $\mathfrak{u}_B$.

Step 2: Since the Chern connection preserves the Hermitian structure g, one has $\nabla(g) = 0$, which gives $\nabla^2(g) = 0$. This means that $\Theta_B \in \Lambda^2 M \otimes \mathfrak{u}_B$, **and this for is real with respect to the real structure defined by ι.**

Step 3: The $(0,2)$-part of the curvature vanishes, because $\overline{\partial}^2 = 0$. The $(2,0)$-part of the curvature vanishes, because $\iota(\Theta_B) = \Theta_B$, and **any real structure on $\operatorname{End}(B)$ exchanges $\Lambda^{2,0}(M) \otimes \operatorname{End}(B)$ and $\Lambda^{0,2}(M) \otimes \operatorname{End}(B)$.**

■

COROLLARY: For the Chern connection $\nabla = \overline{\partial} + \nabla^{1,0}$ on B, one has $\Theta_B = \{\nabla^{1,0}, \overline{\partial}\}$.

COROLLARY: A curvature of a holomorphic Hermitian line bundle is a closed (1,1)-form.

Curvature of the Chern connection on a line bundle

REMARK: Let B he a Hermitian holomorphic line bundle, and $b \in \Gamma(B)$ a nowhere vanishing holomorphic section. Then

$$d|b|^2 = (\nabla^{1,0}b, b) + (b, \nabla^{1,0}b) = 2\operatorname{Re}(\nabla^{1,0}b, b),$$

which gives

$$\nabla^{1,0}b = \frac{\partial |b|^2}{|b|^2} b = 2\partial \log |b| b.$$

We obtaim that $\Theta_B(b) = 2\bar\partial\partial \log |b| b$, hence $\Theta_B = -2\partial\bar\partial \log |b|$.

Corollary 1: Let $g' = e^{2f}g$ be Hermitian metrics on a holomorphic line bundle, and Θ, Θ' the corresponding curvatures. **Then $\Theta' - \Theta = -2\partial\bar\partial f$.** ■

DEFINITION: Tensor multiplication **defines the structure of abelian group on the set $\operatorname{Pic}(M)$ of equivalence classes of holomorphic line bundles on a complex manifod M.** This group is called **the Picard group** of M.

dd^c-lemma (reminder)

THEOREM: Let η be a form on a compact Kähler manifold, satisfying one of the following conditions.
(1). η is an exact (p,q)-form. (2). η is d-exact, d^c-closed.
(3). η is ∂-exact, $\overline{\partial}$-closed.
Then $\eta \in \operatorname{im} dd^c = \operatorname{im} \partial\overline{\partial}$.

Proof: Notice immediately that in all three cases η **is closed and orthogonal to the kernel of Δ**, hence its cohomology class vanishes.

Since η is exact, it lies in the image of Δ. Operator $G_\Delta := \Delta^{-1}$ is defined on $\operatorname{im} \Delta = \ker \Delta^\perp$ and commutes with d, d^c.

In case (1), η is d-exact, and $I(\eta) = \overline{\eta}$ is d-closed, hence η is d-exact, d^c-closed like in (2).

Then $\eta = d\alpha$, where $\alpha := G_\Delta d^* \eta$. Since G_Δ and d^* commute with d^c, the form α is d^c-closed; since it belongs to $\operatorname{im} \Delta = \operatorname{im} G_\Delta$, it is d^c-exact, $\alpha = d^c\beta$ which gives $\eta = dd^c\beta$.

In case (3), we have $\eta = \partial\alpha$, where $\alpha := G_\Delta \partial^* \eta$. Since G_Δ and ∂^* commute with $\overline{\partial}$, the form α is $\overline{\partial}$-closed; since it belongs to $\operatorname{im} \Delta$, it is $\overline{\partial}$-exact, $\alpha = \overline{\partial}\beta$ which gives $\eta = \partial\overline{\partial}\beta$. ■

Forms realized as a curvature of a line bundle

COROLLARY: Let ω be an integer (1,1)-form with integer cohomology class on a compact Kähler manifold. **Then ω is a curvature of a holomorphic line bundle**.

Proof. Step 1: Exponential exact sequence $0 \longrightarrow \mathbb{Z}_M \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_M^* \longrightarrow 0$ gives

$$H^1(\mathcal{O}_M^*) \xrightarrow{c} H^2(M,\mathbb{Z}) \xrightarrow{p} H^2(M,\mathcal{O}_M),$$

where $H^1(\mathcal{O}_M^*) = \operatorname{Pic}(M)$ is the group of holomorphic line bundles, c maps a bundle to its first Chern class, and p projects $H^2(M)$ to its Hodge component $H^2(M,\mathcal{O}_M) = H^{0,2}(M)$. Then **for any integer class $[\omega] \in H^{1,1}(M) \cap H^2(M,\mathbb{Z})$, there exists a line bundle L such that $[\omega] = c_1(L)$.**

Step 2: Take any metric h on L. Its curvature ω_h is a closed (1,1)-form, cohomologous to ω. By dd^c-lemma, $\omega_h - \omega = -2\partial\bar\partial f$ for some $f \in C^\infty M$. **By Corollary 1, curvature of $h' := e^{2f}h$ satisfies $\omega_h - \omega_{h'} = -2\partial\bar\partial f$,** giving $\omega_{h'} = \omega$. ■