Hodge theory

Lecture 21: Hodge theory with coefficients in a bundle and Serre's duality

NRU HSE, Moscow

Misha Verbitsky, April 11, 2018

Curvature (reminder)

DEFINITION: Let $\nabla: B \longrightarrow B \otimes \Lambda^1 M$ be a connection on a vector bundle B. We extend ∇ to an operator

$$V \xrightarrow{\nabla} \Lambda^1(M) \otimes V \xrightarrow{\nabla} \Lambda^2(M) \otimes V \xrightarrow{\nabla} \Lambda^3(M) \otimes V \xrightarrow{\nabla} ...$$

using the Leibnitz identity $\nabla(\eta \otimes b) = d\eta + (-1)^{\tilde{\eta}} \eta \wedge \nabla b$. Then the operator $\nabla^2: B \longrightarrow B \otimes \Lambda^2(M)$ is called **the curvature** of ∇.

REMARK: The algebra of differential forms with coefficients in $\operatorname{End} B$ **acts on** $\Lambda^* M \otimes B$ via $\eta \otimes a(\eta' \otimes b) = \eta \wedge \eta' \otimes a(b)$, where $a \in \operatorname{End}(B)$, $\eta, \eta' \in \Lambda^* M$, and $b \in B$.

REMARK: $\nabla^2(fb) = d^2 fb + df \wedge \nabla b - df \wedge \nabla b + f\nabla^2 b$, hence **the curvature is a** $C^\infty M$**-linear operator. We shall consider the curvature** B **as a 2-form with values in** $\operatorname{End} B$. Then $\nabla^2 := \Theta_B \in \Lambda^2 M \otimes \operatorname{End} B$, where an $\operatorname{End}(B)$-valued form acts on $\Lambda^* M \otimes B$ as above.

Holomorphic bundles (reminder)

DEFINITION: **Holomorphic vector bundle** on a complex manifold M is a locally trivial sheaf of $\mathcal{O}_M$-modules.

DEFINITION: **The total space** $\operatorname{Tot}(B)$ of a holomorphic bundle B over M is the space of all pairs $\{x \in M, b \in B_x/\mathfrak{m}_x B\}$, where B_x is the stalk of B in $x \in M$ and $\mathfrak{m}_x$ the maximal ideal of x. We equip $\operatorname{Tot}(B)$ with the natural topology and holomorphic structure, in such a way that $\operatorname{Tot}(B)$ becomes a locally trivial holomorphic fibration with fiber $\mathbb{C}^r$, $r = \operatorname{rk} B$.

REMARK: **The set of holomorphic sections of a map $\operatorname{Tot}(B) \longrightarrow M$ is naturally identified with the set of sections of the sheaf B.**

CLAIM: Ler B be a holomorphic bundle. Consider the sheaf $B_{C^\infty} := B \otimes_{\mathcal{O}_M} C^\infty M$. **Then B_{C^∞} is a locally trivial sheaf of $C^\infty M$-modules**.

DEFINITION: B_{C^∞} is called **smooth vector bundle underlying the holomorphic vector bundle B**.

REMARK: The natural map $\operatorname{Tot}(B) \longrightarrow \operatorname{Tot}(B_{C^\infty})$ is a diffeomorphism.

$\overline{\partial}$-operator on vector bundles (reminder)

DEFINITION: Let B be a holomorphic vector bundle on M. Consider an operator $\overline{\partial}: B_{C^\infty} \longrightarrow B_{C^\infty} \otimes \Lambda^{0,1}(M)$ mapping $b \otimes f$ to $b \otimes \overline{\partial} f$, where b is a holomorphic section of B, and f smooth. This operator is called **a holomorphic structure operator** on B. **It is well-defined because $\overline{\partial}$ is $\mathcal{O}_M$-linear**, and $B_{C^\infty} = B \otimes_{\mathcal{O}_M} C^\infty M$.

DEFINITION: A **$\overline{\partial}$-operator** on a smooth complex vector bundle V over a complex manifold is a differential operator $V \xrightarrow{\overline{\partial}} \Lambda^{0,1}(M) \otimes V$ satisfying $\overline{\partial}(fb) = \overline{\partial}(f) \otimes b + f\overline{\partial}(b)$ for any $f \in C^\infty M, b \in V$.

REMARK: A $\overline{\partial}$-operator **can be extended to $\overline{\partial}: \Lambda^{0,i}(M) \otimes V \longrightarrow \Lambda^{0,i+1}(M) \otimes V$**, using the Leibnitz identity $\overline{\partial}(\eta \otimes b) = \overline{\partial}(\eta) \otimes b + (-1)^{\tilde{\eta}} \eta \wedge \overline{\partial}(b)$, for all $b \in V$ and $\eta \in \Lambda^{0,i}(M)$.

THEOREM: (Malgrange) Let $\overline{\partial}: V \longrightarrow \Lambda^{0,1}(M) \otimes V$ be a $\overline{\partial}$-operator on a complex vector bundle, satisfying $\overline{\partial}^2 = 0$, where $\overline{\partial}$ is extended to

$$V \xrightarrow{\overline{\partial}} \Lambda^{0,1}(M) \otimes V \xrightarrow{\overline{\partial}} \Lambda^{0,2}(M) \otimes V \xrightarrow{\overline{\partial}} \Lambda^{0,3}(M) \otimes V \xrightarrow{\overline{\partial}} \ldots$$

as above. **Then $B := \ker \overline{\partial} \subset V$ is a holomorphic bundle of the same rank, and $V = B_{\mathbb{C}^\infty}$.**

Chern connection (reminder)

DEFINITION: Let V be a smooth complex vector bundle with connection $\nabla: V \longrightarrow \Lambda^1(M) \otimes V$ and holomorphic structure $\bar\partial: V \longrightarrow \Lambda^{0,1}(M) \otimes V$. Consider the Hodge type decomposition of ∇, $\nabla = \nabla^{0,1} + \nabla^{1,0}$, where

$$\nabla^{0,1}: V \longrightarrow \Lambda^{0,1}(M) \otimes V, \quad \nabla^{1,0}: V \longrightarrow \Lambda^{1,0}(M) \otimes V.$$

We say that **the connection ∇ is compatible with the holomorphic structure** if $\nabla^{0,1} = \bar\partial$.

DEFINITION: A holomorphic Hermitian vector buncle is a smooth complex vector bundle equipped with a Hermitian metric and a holomorphic structure.

DEFINITION: Chern connection on a holomorphic Hermitian vector bundle is a unitary connection compatible with the holomorphic structure.

THEOREM: Every holomorphic Hermitian vector bundle **admits a Chern connection, which is unique.**

REMARK: When people say about "curvature of a holomorphic Hermitian line bundle", **they speak about curvature of a Hermitian connection.**

Curvature of Chern connection (reminder)

REMARK: $[d_\nabla, \{d_\nabla, d_\nabla\}] = [\{d_\nabla, d_\nabla\}, d_\nabla] + [d_\nabla, \{d_\nabla, d_\nabla\}] = 0$ by the super Jacobi identity. This gives **the Bianchi identity:** $d_\nabla(\Theta_B \wedge \eta) = \Theta_B \wedge d_\nabla(\eta)$, giving $d_\nabla(\Theta_B) = 0$.

REMARK: When B is a line bundle, $\operatorname{End}(B)$ is trivial, and Θ_B is a 2-form. **In this case $d_\nabla = \nabla$ and Θ_B is closed.**

DEFINITION: When B is a line bundle, the cohomology class of $-\frac{\sqrt{-1}}{\pi}\Theta_B$ is called **first Chern class of B**, denoted $c_1(B)$.

THEOREM: The exponential exact sequence $0 \longrightarrow \mathbb{Z}_M \longrightarrow C^\infty M \longrightarrow (C^\infty M)^* \longrightarrow 0$ gives an isomorphism $H^1(M, (C^\infty M)^*) \xrightarrow{c_1^{\mathbb{Z}}} H^2(M, \mathbb{Z})$. Let L be a line bundle associated with a cocycle η. **Then the image of $c_1^{\mathbb{Z}}(\eta)$ in $H^2(M, \mathbb{R})$ is $c_1(L)$.**

Supersymmetry in Kähler geometry (reminder)

Let (M, I, g) be a Kaehler manifold, ω its Kaehler form. **On $\Lambda^*(M)$, the following operators are defined.**

0. d, d^*, Δ, because it is Riemannian.
1. $L(\alpha) := \omega \wedge \alpha$, $\Lambda(\alpha) := *L * \alpha$. It is easily seen that $\Lambda = L^*$.
3. The Weil operator $W\Big|_{\Lambda^{p,q}(M)} = \sqrt{-1}\,(p-q)$

THEOREM: (Kähler package)
These operators generate a Lie superalgebra $\mathfrak{a}$ of dimension $(5|4)$, acting on $\Lambda^*(M)$. Moreover, the Laplacian Δ is central in $\mathfrak{a}$, hence $\mathfrak{a}$ **also acts on the cohomology of M.**

The odd part of this algebra generates **an odd Heisenberg superalgebra** $\langle d, d^c, d^*, (d^c)^*, \Delta\rangle$, with the only non-zero anticommutator $\{d, d^*\} = \{d^c, (d^c)^*\} = \Delta$.

The even part of this algebra contains an $\mathfrak{sl}(2)$-triple $\langle L, \Lambda, H\rangle$ acting on $\mathfrak{a}^{\text{odd}}$ as on a direct sum of two weight 1 representations ("Kodaira relations"). The Weil element commutes with $\langle L, \Lambda, H, \Delta\rangle$ and acts on $\mathfrak{a}^{\text{odd}}$ via $[W, d] = d^c$, $[W, d^*] = (d^c)^*$.

Coordinate operators (reminder)

Let V be an even-dimensional real vector space equipped with a scalar product, and $v_1, ..., v_{2n}$ an orthonormal basis. Denote by $e_{v_i}: \Lambda^k V \longrightarrow \Lambda^{k+1} V$ an operator of multiplication, $e_{v_i}(\eta) = v_i \wedge \eta$. Let $i_{v_i}: \Lambda^k V \longrightarrow \Lambda^{k-1} V$ be an adjoint operator, $i_{v_i} = * e_{v_i} *$.

CLAIM: The operators e_{v_i}, i_{v_i}, Id are a basis of an **odd Heisenberg Lie superalgebra** $\mathfrak{H}$, with **the only non-trivial supercommutator given by the formula** $\{e_{v_i}, i_{v_j}\} = \delta_{i,j} \operatorname{Id}$.

Now, consider the tensor $\omega = \sum_{i=1}^n v_{2i-1} \wedge v_{2i}$, and let $L(\alpha) = \omega \wedge \alpha$, and $\Lambda := L^*$ be the corresponding **Hodge operators**.

REMARK: Relations in $\mathfrak{H}$ imply that $H := [L, \Lambda] = \left[\sum e_{v_{2i-1}} e_{v_{2i}}, \sum i_{v_{2i-1}} i_{v_{2i}}\right] = \sum_{i=1}^{2n} e_{v_i} i_{v_i} - \sum_{i=1}^{2n} i_{v_i} e_{v_i}$ **is the scalar operator acting on k-forms as multiplication by $n-k$.**

COROLLARY: The triple L, Λ, H satisfies the relations for the standard generators of $\mathfrak{sl}(2)$: $[L, \Lambda] = H$, $[H, L] = 2L$, $[H, \Lambda] = -2\Lambda$.

Kodaira identities (reminder)

THEOREM: Let M be a Kähler manifold. One has the following identities **("Kähler idenitities", "Kodaira idenities").**

$$[\Lambda, \partial] = \sqrt{-1}\,\overline{\partial}^*, \quad [L, \overline{\partial}] = -\sqrt{-1}\,\partial^*, \quad [\Lambda, \overline{\partial}^*] = -\sqrt{-1}\,\partial, \quad [L, \partial^*] = \sqrt{-1}\,\overline{\partial}.$$

Equivalently,

$$[\Lambda, d] = (d^c)^*, \qquad [L, d^*] = -d^c, \qquad [\Lambda, d^c] = -d^*, \qquad [L, (d^c)^*] = d.$$

Proof. Step 1: The first set of identities implies the second set. Indeed, by adding up appropriate identities in the top set of their complex conjugate, we obtain ones in the bottom set; for example, adding $[\Lambda, \partial] = \sqrt{-1}\,\overline{\partial}^*$ and $[\Lambda, \overline{\partial}] = -\sqrt{-1}\,\partial^*$,, we obtain $[\Lambda, d] = (d^c)^*$. Each of top identities is related to the other three by complex conjugation or by Hermitian conjugation, hence they are all equivalent. Each of the bottom identities implies the rest by Hermitian conjugation and conjugating with I. Finally, $[\Lambda, \partial] = \sqrt{-1}\,\overline{\partial}^*$ can be obtained as a sum of $[\Lambda, d] = (d^c)^*$ and $[\Lambda, d^c] = -d^*$ with appropriate coefficients. **We obtained that all Kodaira identities are implied by just one, say, $[L, d^*] = -d^c$.**

Kodaira identities 2 (reminder)

Proof. Step 1: **We reduced the Kodaira identities to just one, $[L, d^*] = -d^c$.**

Step 2: Let $\mathfrak{E} : \Lambda^i M \otimes \Lambda^1 M \longrightarrow \Lambda^{i+1}(M)$ be the multiplication, and $\mathfrak{I} : \Lambda^i M \otimes \Lambda^1 M \longrightarrow \Lambda^{i-1}(M)$ the map that takes $\alpha \wedge \theta$ and puts it to $*(*\alpha \wedge \theta)$. In other words, $\mathfrak{I}$ takes a tensor $\alpha \otimes \theta$, with $\alpha \in \Lambda^i M$ and $\theta \in \Lambda^1 M$, uses the metric g to produce a vector field X from θ, and maps α to $\alpha \lrcorner X$ (convution of α and X).

Step 3: Let ∇ be the Levi-Civita connection. Then $d\alpha = \mathfrak{E}(\nabla(\alpha))$, because ∇ is torsion-free. Since $d^* = *d*$, one has $d^*(\alpha) = \mathfrak{I}(\nabla(\alpha))$. Let $x_1, y_1, ..., x_n, y_n \in \Lambda^1_m M$ be an orthonormal basis such that $\omega = \sum x_i \wedge y_i$. Then $\mathfrak{I}(\nabla(\alpha)) = \sum_i i_{x_i}(\nabla_{x_i}\alpha) + i_{y_i}(\nabla_{y_i}\alpha)$. Taking a commutator with $L = \sum e_{x_i}e_{y_i}$ and using the commutator relations between e_v and i_w found earlier, we obtain

$$[L, d^*] = \sum_i \nabla_{x_i}[e_{x_i}e_{y_i}, i_{x_i}] + \nabla_{y_i}[e_{x_i}e_{y_i}, i_{y_i}] = \sum_i \nabla_{y_i}e_{x_i} - \nabla_{x_i}e_{y_i}.$$

(the operator ∇_w commutes with L, because ω is parallel). However,

$$\sum_i \nabla_{y_i}e_{x_i} - \nabla_{x_i}e_{y_i} = -I\left(\sum_i \nabla_{x_i}e_{x_i} + \nabla_{y_i}e_{y_i}\right) = -d^c$$

which gives $[L, d^*] = -d^c$. ■

Conventions

Let B be a holomorphic Hermitian bundle, and $\nabla = \overline{\partial} + \nabla^{1,0}$ its Chern connection. In this lecture, **I would use ∂ instead of $\nabla^{1,0}$.**

As usual, we define the sequence

$$V \xrightarrow{\nabla} \Lambda^1(M) \otimes V \xrightarrow{\nabla} \Lambda^2(M) \otimes V \xrightarrow{\nabla} \Lambda^3(M) \otimes V \xrightarrow{\nabla} ...$$

using the Leibnitz identity $\nabla(\eta \otimes b) = d\eta + (-1)^{\tilde{\eta}}\eta \wedge \nabla b$.

Here the operators ∇ are denoted by d_∇, their (1,0) and (0,1)-parts ∂_∇ and $\overline{\partial}_\nabla$. **To simplify notations, I shall often omit the subscript $_\nabla$.**

These conventions are sloppy but more or less standard.

Kodaira relations

PROPOSITION: Let B be a holomorphic Hermitian bundle on a Kähler manifold, and $\nabla = \overline{\partial} + \partial$ its Chern connection. **On B-valued differential forms, the following relations hold.**

$$[\Lambda, \partial] = \sqrt{-1}\, \overline{\partial}^*, \quad [\Lambda, \overline{\partial}] = -\sqrt{-1}\, \partial^*, \quad [L, \overline{\partial}^*] = -\sqrt{-1}\, \partial, \quad [L, \partial^*] = \sqrt{-1}\, \overline{\partial}.$$

Proof: It suffices to show, for example, that $[L, \partial^*] = -\sqrt{-1}\, \overline{\partial}$. Let $\mathfrak{E} : \Lambda^i M \otimes B \otimes \Lambda^1 M \longrightarrow \Lambda^{i+1}(M) \otimes B$ denote the multiplication map, and $\mathfrak{I} : \Lambda^i M \otimes B \otimes \Lambda^1 M \longrightarrow \Lambda^{i-1}(M) \otimes B$ denote the convolution with the dual vector field. **Then $\overline{\partial}(\eta) = \mathfrak{E}^{0,1}(\overline{\partial}(\eta))$, and $\partial^*(\eta) = \mathfrak{I}^{-1,0}(\overline{\partial}(\eta))$.** Here $\overline{\partial}$, ∂ are $(0,1)$ and $(1,0)$-parts of the connection in $\Lambda^*(M) \otimes B$ induced from the Levi-Civita connection on $\Lambda^*(M)$ and the Chern connection on B.

Step 2: Since ∇ commutes with L, we have $[L, \partial^*] = [L, \mathfrak{I}^{-1,0}] \circ \nabla$, where $L: \Lambda^i M \otimes \Lambda^1 M \otimes B \longrightarrow \Lambda^{i+2} M \otimes \Lambda^1 M \otimes B$ acts on the first component. Taking an orthonormal basis z_i in $\Lambda^{1,0}(M)$, we obtain $L = -\sqrt{-1} \sum e_{z_i} e_{\overline{z}_i}$. Using the relations in the odd Heisenberg algebra, we obtain $[e_x, \mathfrak{I}] = 1$. This gives $[L, \mathfrak{I}^{-1,0}](z_i \otimes \eta) = -\sqrt{-1}\, e_{\overline{z}_i}$ $[L, \mathfrak{I}^{-1,0}](\overline{z}_i \otimes \eta) = 0$, hence **$[L, \mathfrak{I}^{-1,0}](a \otimes \eta) = -\sqrt{-1}\, \mathfrak{E}^{0,1}(a \otimes \eta)$.**

Step 3: Comparing the results of step 1 and 2, $\overline{\partial}(\eta) = \mathfrak{E}^{0,1}(\overline{\partial}(\eta))$, and $\partial^*(\eta) = \mathfrak{I}^{-1,0}(\overline{\partial}(\eta))$ with $[L, \mathfrak{I}^{-1,0}](a \otimes \eta) = -\sqrt{-1}\, \mathfrak{E}^{0,1}(a \otimes \eta)$, we obtain $[L, \partial^*] = -\sqrt{-1}\, \overline{\partial}$. ■

Hodge theory with coefficients in a bundle

DEFINITION: Let $\overline{\partial}: \Lambda^{p,q}(M)\otimes B \longrightarrow \Lambda^{p,q+1}(M)\otimes B$ be the holomorphic structure operator, extended to differential forms using the Leibniz identity. The anticommutator $\Delta_{\overline{\partial}} := \{\overline{\partial}, \overline{\partial}^*\} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ is called **Dolbeault Laplacian with coefficients in B**. It is self-adjoint, positive elliptic operator: $(\Delta_{\overline{\partial}}x, x) = (\overline{\partial}x, \overline{\partial}x) + (\overline{\partial}^*x, \overline{\partial}^*x)$.

Hodge theory with coefficients in a bundle **There is a basis in the Hilbert space $L^2(\Lambda^*(M)\otimes B)$, consisting of eigenvalues of $\Delta_{\overline{\partial}}$, and each eigenspace is finite-dimensional.**

Elliptic regularity **Each eigenvector of $\Delta_{\overline{\partial}}$ on $L^2(\Lambda^*(M)\otimes B)$ is a smooth form.**

DEFINITION: A B-valued form η is called **$\Delta_{\overline{\partial}}$-harmonic** if $\eta \in \Delta_{\overline{\partial}}$.

REMARK: If A, B holomorphic vector bundles, $A\otimes B$ means $A\otimes_{\mathcal{O}M} B$. If A is smooth, B is holomorphic, $A\otimes B$ means $A\otimes_{\mathcal{O}M} B$. If both A and B are smooth bundles, $A\otimes B$ means $A\otimes_{C^\infty M} B$. **The ranks of the bundles are multiplied in all three cases.**

Dolbeault cohomology with coefficients in a bundle

THEOREM: The space of $\Delta_{\overline{\partial}}$-harmonic forms is identified with the **Dolbeault cohomology with coefficients in B**, that is, with $\frac{\ker \overline{\partial}}{\operatorname{im} \overline{\partial}}$. We denote the (p,q)-part of Dolbeault cohomology by $H^{p,q}(M,B)$.

Proof: Since $\Delta_{\overline{\partial}}$ is elliptic, $\Lambda^*(M) = \operatorname{im} \Delta_{\overline{\partial}} \oplus \ker \Delta_{\overline{\partial}}$. This gives an orthogonal decomposition $\Lambda^*(M) = \operatorname{im} \overline{\partial} \oplus \ker \Delta_{\overline{\partial}} \oplus \operatorname{im} \overline{\partial}^*$. Since $\operatorname{im} \overline{\partial}^* = (\ker \overline{\partial})^\perp$, this decomposition identifies $\ker \Delta_{\overline{\partial}}$ with cohomology of $\overline{\partial}$. ∎

CLAIM: The cohomology $H^{p,q}(M,B)$ are identified with the $H^q(B \otimes \Omega^p M)$, where $\Omega^p M$ is the sheaf of holomorphic p-forms.

Proof: The sequence of sheaves

$$0 \longrightarrow \Omega^p M \otimes B \hookrightarrow \Lambda^{p,0} M \otimes B \xrightarrow{\overline{\partial}} \Lambda^{p,1} M \otimes B \overline{\partial} \longrightarrow \Lambda^{p,2} M \otimes B$$

is exact by Poincaré-Dolbeault-Grothendieck lemma, hence it gives an acyclic resolution of $\Omega^p(M) \otimes B$, and cohomology of its global sections are identified with the cohomology of the sheaf $B \otimes \Omega^p M$. ∎

Serre's duality

REMARK: The operator $* : \Lambda^{p,i}(M) \otimes_{\mathcal{O}_M} B \longrightarrow \Lambda^{n-p,n-i}(M) \otimes_{\mathcal{O}_M} B^*$ exchanges $\overline{\partial}$ and $\pm\overline{\partial}^*$. **Therefore, it preserves the space** $\ker \Delta_{\overline{\partial}}$ of $\overline{\partial}$-harmonic forms.

REMARK: By definition, $H^n(\Omega^n M) = \frac{\Lambda^{n,n}(M)}{\operatorname{im}\overline{\partial}}$. Stokes formula implies that $\int_M \overline{\partial}(\alpha) = 0$ for all α. This gives a natural map $H^n(\Omega^n M) \xrightarrow{\int_M} \mathbb{C}$.

THEOREM: (Serre's duality)
Let M be an n-dimensional, compact complex manifold, and B an Hermitian holomorphic bundle. Then **the multiplication**

$$H^i(\Omega^p M \otimes B) \times H^{n-i}(\Omega^{n-p} M \otimes B^*) \longrightarrow H^n(\Omega^n M) \xrightarrow{\int_M} \mathbb{C}$$

defines a non-degenerate pairing.

Proof: Indeed, for each $\eta \in \ker \Delta_{\overline{\partial}}$, the form $*\eta$ also belongs to $\ker \Delta_{\overline{\partial}}$, but $\int_M \eta \wedge *\eta > 0$, hence this pairing is non-degenerate. ■

COROLLARY: (Serre's duality for $p = n$) Let M be a compact complex manifold, $\dim_{\mathbb{C}} M = n$. **Then $H^i(B) \cong H^{n-i}(B^* \otimes K_M)^*$, where $K_M = \Omega^n M$.**

DEFINITION: $K_M = \Omega^n M$ is called **canonical bundle**.

Schedule for April and May

Lectures: Misha Verbitsky (April 14, May 16)
Ekaterina Amerik (April 18, 21, 25)

Exam: Saturday, May 19.