Hodge theory

Lecture 22: Kodaira-Nakano vanishing theorem

NRU HSE, Moscow

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Curvature (reminder)

DEFINITION: Let $\nabla: B \longrightarrow B \otimes \Lambda^1 M$ be a connection on a vector bundle B. We extend ∇ to an operator

$$V \xrightarrow{\nabla} \Lambda^1(M) \otimes V \xrightarrow{\nabla} \Lambda^2(M) \otimes V \xrightarrow{\nabla} \Lambda^3(M) \otimes V \xrightarrow{\nabla} ...$$

using the Leibnitz identity $\nabla(\eta \otimes b) = d\eta + (-1)^{\tilde\eta}\eta \wedge \nabla b$. Then the operator $\nabla^2: B \longrightarrow B \otimes \Lambda^2(M)$ is called **the curvature** of ∇.

REMARK: The algebra of differential forms with coefficients in $\operatorname{End} B$ **acts on** $\Lambda^* M \otimes B$ via $\eta \otimes a(\eta' \otimes b) = \eta \wedge \eta' \otimes a(b)$, where $a \in \operatorname{End}(B)$, $\eta, \eta' \in \Lambda^* M$, and $b \in B$.

REMARK: $\nabla^2(fb) = d^2 fb + df \wedge \nabla b - df \wedge \nabla b + f\nabla^2 b$, hence **the curvature is a** $C^\infty M$**-linear operator. We shall consider the curvature** B **as a 2-form with values in** $\operatorname{End} B$. Then $\nabla^2 := \Theta_B \in \Lambda^2 M \otimes \operatorname{End} B$, where an $\operatorname{End}(B)$-valued form acts on $\Lambda^* M \otimes B$ as above.

Conventions

Let B be a holomorphic Hermitian bundle, and $\nabla = \overline{\partial} + \nabla^{1,0}$ its Chern connection. In this lecture, **I would use ∂ instead of $\nabla^{1,0}$.**

As usual, we define the sequence

$$V \xrightarrow{\nabla} \Lambda^1(M) \otimes V \xrightarrow{\nabla} \Lambda^2(M) \otimes V \xrightarrow{\nabla} \Lambda^3(M) \otimes V \xrightarrow{\nabla} ...$$

using the Leibnitz identity $\nabla(\eta \otimes b) = d\eta + (-1)^{\tilde{\eta}} \eta \wedge \nabla b$.

Here the operators ∇ are denoted by d_∇, their (1,0) and (0,1)-parts ∂_∇ and $\overline{\partial}_\nabla$. **To simplify notations, I shall often omit the subscript $_\nabla$.**

These conventions are sloppy but more or less standard.

Hodge theory with coefficients in B (reminder)

PROPOSITION: Let B be a holomorphic Hermitian bundle on a Kähler manifold, and $\nabla = \overline{\partial} + \partial$ its Chern connection. We use the same letters ∂, $\overline{\partial}$ for the Hodge components of $d_\nabla : \Lambda^i(M) \otimes B \longrightarrow \Lambda^{i+1}(M) \otimes B$. Then **on B-valued differential forms, the following relations hold.**

$$[\Lambda, \partial] = \sqrt{-1}\, \overline{\partial}^*, \quad [\Lambda, \overline{\partial}] = -\sqrt{-1}\, \partial^*, \quad [L, \overline{\partial}^*] = -\sqrt{-1}\, \partial, \quad [L, \partial^*] = \sqrt{-1}\, \overline{\partial}.$$

DEFINITION: Let $\overline{\partial} : \Lambda^{p,q}(M) \otimes B \longrightarrow \Lambda^{p,q+1}(M) \otimes B$ be the holomorphic structure operator, extended to differential forms using the Leibniz identity. The anticommutator $\Delta_{\overline{\partial}} := \{\overline{\partial}, \overline{\partial}^*\} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ is called **Dolbeault Laplacian with coefficients in B**. It is self-adjoint, positive elliptic operator: $(\Delta_{\overline{\partial}} x, x) = (\overline{\partial} x, \overline{\partial} x) + (\overline{\partial}^* x, \overline{\partial}^* x)$.

Hodge theory with coefficients in a bundle: There is a basis in the Hilbert space $L^2(\Lambda^*(M) \otimes B)$, consisting of eigenvalues of $\Delta_{\overline{\partial}}$, and each eigenspace is finite-dimensional.
Elliptic regularity: Each eigenvector of $\Delta_{\overline{\partial}}$ on $L^2(\Lambda^*(M) \otimes B)$ is a smooth form.

THEOREM: The space of $\Delta_{\overline{\partial}}$-harmonic forms is identified with the **Dolbeault cohomology with coefficients in B**, that is, with $\frac{\ker \overline{\partial}}{\operatorname{im} \overline{\partial}}$. We denote the (p,q)-part of Dolbeault cohomology by $H^{p,q}(M, B)$.

Serre's duality (reminder)

REMARK: The operator $* : \Lambda^{p,i}(M) \otimes_{\mathcal{O}_M} B \longrightarrow \Lambda^{n-p,n-i}(M) \otimes_{\mathcal{O}_M} B^*$ exchanges $\bar\partial$ and $\pm\bar\partial^*$. **Therefore, it preserves the space** $\ker \Delta_{\bar\partial}$ of $\bar\partial$-harmonic forms.

REMARK: By definition, $H^n(\Omega^n M) = \frac{\Lambda^{n,n}(M)}{\operatorname{im}\bar\partial}$. Stokes formula implies that $\int_M \bar\partial(\alpha) = 0$ for all α. This gives a natural map $H^n(\Omega^n M) \xrightarrow{\int_M} \mathbb{C}$.

THEOREM: (Serre's duality)
Let M be an n-dimensional, compact complex manifold, and B an Hermitian holomorphic bundle. Then **the multiplication**

$$H^i(\Omega^p M \otimes B) \times H^{n-i}(\Omega^{n-p} M \otimes B^*) \longrightarrow H^n(\Omega^n M) \xrightarrow{\int_M} \mathbb{C}$$

defines a non-degenerate pairing.

Proof: Indeed, for each $\eta \in \ker \Delta_{\bar\partial}$, the form $*\eta$ also belongs to $\ker \Delta_{\bar\partial}$, but $\int_M \eta \wedge *\eta > 0$, hence this pairing is non-degenerate. ■

COROLLARY: (Serre's duality for $p = n$) Let M be a compact complex manifold, $\dim_{\mathbb{C}} M = n$. **Then $H^i(B) \cong H^{n-i}(B^* \otimes K_M)^*$, where $K_M = \Omega^n M$.**

DEFINITION: $K_M = \Omega^n M$ is called **canonical bundle**.

Schedule for April and May

Lectures: Misha Verbitsky (April 14, May 16)
Ekaterina Amerik (April 18, 21, 25)

Exam: Saturday, May 19.

Laplacians and curvature

REMARK: Curvature of Chern connection: $\Theta_B = \{\nabla^{1,0}, \overline{\partial}\}$, in the present notation it's $\Theta_B = \{\partial, \overline{\partial}\}$.

THEOREM: (Bochner-Kodaira-Nakano identity)
$\Delta_{\overline{\partial}} = \Delta_\partial + H_B$, **where** $H_B := -\sqrt{-1}\,[\Lambda, \Theta_B]$**.**

Proof: Super-Jacobi identity gives

$$[\Lambda, \Theta_B] = [\Lambda, \{\partial, \overline{\partial}\}] = \{[\Lambda, \partial], \overline{\partial}\} + \{\partial, [\Lambda, \overline{\partial}]\}$$
$$= \sqrt{-1}\,\{\overline{\partial}^*, \overline{\partial}\} - \sqrt{-1}\,\{\partial^*, \partial\} = \sqrt{-1}\,\Delta_{\overline{\partial}} - \sqrt{-1}\,\Delta_\partial.$$

■

REMARK: The operators $\Delta_{\overline{\partial}}$ and Δ_∂ are positive. Therefore, if $(H_B x, x) > 0$, one has $(\Delta_{\overline{\partial}} x, x) > 0$. **If $(H_B x, x) > 0$ for all x, the operator $\Delta_{\overline{\partial}}$ has no kernel, and the corresponding cohomology group vanishes.**

Positive line bundles

DEFINITION: A holomorphic line bundle is called **positive** if its first Chern class is cohomologous to a Kähler form.

Theorem 1: (Kodaira-Nakano) Let L be a positive line bundle on a compact Kähler manifold. **Then for any bundle B there exists $N > 0$ such that $H^i(B \otimes L^N) = 0$ for all $i > 0$.**

Proof: We deduce Theorem 1 from Theorem 2 below.

Theorem 2: (Kodaira-Nakano) Let B be a holomorphic Hermitian line bundle on n-dimensional Kähler manifold, Θ_B its curvature, and $L_{\Theta_B}: \Lambda^*(M) \longrightarrow \Lambda^*(M)$ the operator of multiplication by Θ_B. Suppose that the self-adjoint operator $H_B := \sqrt{-1}\,[L_{\Theta_B}, \Lambda]$ satisfies $(H_B(x), x) > 0$ for any non-zero k-form x, $k > n$. **Then $H^p(B \otimes \Omega^q M) = 0$ for all $p + q > n$.**

Proof: $\Delta_{\bar\partial} = \Delta_\partial - H_B$, which gives $(\Delta_{\bar\partial} x, x) = (\Delta_\partial x, x) - (H_B x, x) > 0$. ■

Kunihiko Kodaira

Kunihiko Kodaira (1915-1997)

Negative line bundles and cohomology

REMARK: For any vector bundles E, F with connections, their curvatures are related as $\Theta_{E\otimes F} = \Theta_E + \Theta_F$. Also, if L is a line bundle, one has $\Theta_{L^*} = -\Theta_L$.

DEFINITION: Let L be a line bundle such that L^* is positive. Then L is called **negative**.

REMARK: Suppose that L is a negative line bundle on (M, ω), and $\sqrt{-1}\,\omega$ the curvature of L. Then $H_L := \sqrt{-1}\,[L_{\Theta_L}, \Lambda] = H$. **On k-forms H_L acts as multiplication by $k-n$. Therefore, L satisfies conditions of Theorem 2, and $H^p(L\otimes \Omega^q M) = 0$ for all $p+q > n$.**

Kodaira-Nakano theorem: the proof

REMARK: Suppose L is a negative vector bundle, with $H_L = -\sqrt{-1}[L_{\Theta_L}, \Lambda] = H$. The operator $H_{E\otimes F}$ is expressed as $H_{E\otimes F} = H_E + H_F$. This gives $H_{B\otimes L^N} = H_B - NH$. For $N > \alpha$, where α is the biggest eigenvalue of H_B, we have

$$(H_{B\otimes L^N}x, x) = (H_B x, x) + N(n-k)|x|^2 > 0.$$

Then Theorem 2 gives $H^p(L^N \otimes B \otimes \Omega^q M) = 0$ for all $p + q > n$.

COROLLARY: Let L be a negative line bundle. Then **for all vector bundles B with Chern connection, there exists $N > 0$ such that $H^i(L^{-N} \otimes B) = 0$ for all $i > 0$.**

Proof: Apply the previous corollary to $B^* \otimes K_M$, and use the Serre's duality

$$0 = H^{n-i}(L^N \otimes B^* \otimes \Omega^m M) = H^i(B \otimes L^{-N})^*.$$

■

REMARK: We have proved Theorem 1. ■

Forms realized as a curvature (reminder)

Proposition 1: Let ω be an integer (1,1)-form with integer cohomology class on a compact Kähler manifold. **Then ω is a curvature of a holomorphic line bundle**.

Proof. Step 1: Exponential exact sequence $0 \longrightarrow \mathbb{Z}_M \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_M^* \longrightarrow 0$ gives

$$H^1(\mathcal{O}_M^*) \xrightarrow{c} H^2(M,\mathbb{Z}) \xrightarrow{p} H^2(M,\mathcal{O}_M),$$

where $H^1(\mathcal{O}_M^*) = \operatorname{Pic}(M)$ is the group of holomorphic line bundles, c maps a bundle to its first Chern class, and p projects $H^2(M)$ to its Hodge component $H^2(M,\mathcal{O}_M) = H^{0,2}(M)$. Then **for any integer class $[\omega] \in H^{1,1}(M) \cap H^2(M,\mathbb{Z})$, there exists a line bundle L such that $[\omega] = c_1(L)$.**

Step 2: Take any metric h on L. Its curvature ω_h is a closed (1,1)-form, cohomologous to ω. By dd^c-lemma, $\omega_h - \omega = -2\partial\overline{\partial} f$ for some $f \in C^\infty M$. **By Corollary 1, curvature of $h' := e^{2f}h$ satisfies $\omega_h - \omega_{h'} = -2\partial\overline{\partial} f$,** giving $\omega_{h'} = \omega$. ∎

Base points of line bundles

DEFINITION: Let L be a holomorphic line bundle on M. A point $x \in M$ is called **base point** for M if any global section of L **vanishes in x,** that is, if it does not generate the stalk L_x over the ring of germs $\mathcal{O}_{M,x}$.

A bundle is called **base point free** if it has no base points.

EXAMPLE: Let $\mathbb{C}^{n+1}\backslash 0 \xrightarrow{\pi} \mathbb{C}P^n$ be the natural projection. For each point $x \in \mathbb{C}P^n$, the fiber $\pi^{-1}(x)$ is $\mathbb{C}^* \subset l_x$, where l_x is the line associated with x. Denote by $\mathcal{O}(-1)$ the bundle with the fiber l_x in each $x \in \mathbb{C}P^n$. Each line functional on $\mathbb{C}^{n+1}$ defines a line functional on each l_x, which holomorphically depends on x. **This means that the dual line bundle, denoted $\mathcal{O}(1)$, is base point free.**

REMARK: Let L be a base point free line bundle, and $\mathfrak{m}_x \subset \mathcal{O}_{M,x}$ the maximal ideal of a point $x \in M$. Consider **the fiber** $L|_x := L_x/\mathfrak{m}_x$. It is a vector space over $\mathbb{C}$ of rank 1. A map $H^0(M,L) \longrightarrow L_x$ taking f to $f|_x$ defines a map $\varphi_x \in \operatorname{Hom}_{\mathbb{C}}(H^0(M,L), L|_x) \cong H^0(M,L)^*$. **The isomorphism $\operatorname{Hom}_{\mathbb{C}}(H^0(M,L), L|_x) \cong H^0(M,L)^*$ is defined only after we fix an isomorphism $L|_x \cong \mathbb{C}$,** that is, up to a constant multiplier. Therefore, **the map $x \longrightarrow \psi_x$ defines a holomorphic map $\varphi: M \longrightarrow \mathbb{P}H^0(M,L)^*$.** **This map is well-defined only if L is base point free.**

Very ample line bundles

CLAIM: Let L be a base point free holomorhic line bundle on M, and $\varphi : M \longrightarrow \mathbb{P}H^0(M,L)^*$ the natural map defined above. **Then $L \cong \varphi^*(\mathcal{O}(1))$.**

Proof. Step 1: This statement is essentially a tautology: global sections of $\mathcal{O}(1)$ on $\mathbb{P}V$ are identified with V^*, hence global sections of $\varphi^*(\mathcal{O}(1))$ are identified with $H^0(M,L)$.

Step 2: To see that this identification is carried over to the corresponding sheaves, we consider the tautological line bundle $\mathcal{O}(-1)$ with the fiber l_x at each $x \in \mathbb{P}H^0(M,L)^*$. By construction, l_x is dual to $L|_x$, and this duality is holomorphic on x, hence $\varphi^*(\mathcal{O}(-1)) = L^*$. ∎

DEFINITION: A line bundle L is called **very ample** if it is base point free, and the corresponding holomorphic map $\varphi : M \longrightarrow \mathbb{P}H^0(M,L)^*$ is an embedding. It is called **ample** of $L^{\otimes N}$ is very ample for some $N > 0$.

DEFINITION: A complex manifold is called **projective** if it admits a holomorphic embedding to $\mathbb{C}P^n$.

Kodaira embedding theorem

COROLLARY: Let (M,ω) be a compact Kähler manifold such that the cohomology class of ω is rational. **Then M admits a positive line bundle.**

Proof: Let $n\omega$ be a Kähler form proportional to ω with integer Kähler class. **Then $n\omega$ is realized as a curvature of a holomorphic line bundle (Proposition 1).** This line bundle is positive. ■

THEOREM: (Kodaira) A positive bundle is ample.
Proof: *(will be proven later in April)* ■

COROLLARY: (Kodaira embedding theorem)
A compact Kähler manifold M is projective if and only if it admits a Kähler form with rational cohomology class.

Proof: Using the Corollary above, we obtain that M admits a positive holomorphic line bundle L, and the theorem of Kodaira above implies that L is ample, hence $L^{\otimes N}$ defines a projective embedding.

Conversely, if $M \subset \mathbb{C}P^n$ is a projective manifold, the cohomology class of Fubini-Study form is proportional to a rational class, hence M admits a Kähler form with a rational cohomology class. ■