Hodge theory

Lecture 23: Calabi-Yau theorem

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REMINDER: Holomorphic vector bundles

DEFINITION: A $\overline{\partial}$-operator on a smooth bundle is a map $V \xrightarrow{\overline{\partial}} \Lambda^{0,1}(M) \otimes V$, satisfying $\overline{\partial}(fb) = \overline{\partial}(f) \otimes b + f\overline{\partial}(b)$ for all $f \in C^\infty M, b \in V$.

REMARK: A $\overline{\partial}$-operator on B can be extended to

$$\overline{\partial}: \Lambda^{0,i}(M) \otimes V \longrightarrow \Lambda^{0,i+1}(M) \otimes V,$$

using $\overline{\partial}(\eta \otimes b) = \overline{\partial}(\eta) \otimes b + (-1)^{\tilde{\eta}} \eta \wedge \overline{\partial}(b)$, where $b \in V$ and $\eta \in \Lambda^{0,i}(M)$.

DEFINITION: A holomorphic vector bundle on a complex manifold (M, I) is a vector bundle equipped with a $\overline{\partial}$-operator which satisfies $\overline{\partial}^2 = 0$. In this case, $\overline{\partial}$ is called **a holomorphic structure operator**.

EXERCISE: Consider the Dolbeault differential $\overline{\partial}: \Lambda^{p,0}(M) \longrightarrow \Lambda^{p,1}(M) = \Lambda^{p,0}(M) \otimes \Lambda^{0,1}(M)$. **Prove that it is a holomorphic structure operator on $\Lambda^{p,0}(M)$.**

DEFINITION: The corresponding holomorphic vector bundle $(\Lambda^{p,0}(M), \overline{\partial})$ is called **the bundle of holomorphic p-forms**, denoted by $\Omega^p(M)$.

REMINDER: Chern connection

DEFINITION: Let (B, ∇) be a smooth bundle with connection and a holomorphic structure $\bar\partial\ B \longrightarrow \Lambda^{0,1}(M) \otimes B$. Consider a Hodge decomposition of ∇, $\nabla = \nabla^{0,1} + \nabla^{1,0}$,

$$\nabla^{0,1}:\ V \longrightarrow \Lambda^{0,1}(M) \otimes V, \quad \nabla^{1,0}:\ V \longrightarrow \Lambda^{1,0}(M) \otimes V.$$

We say that ∇ is **compatible with the holomorphic structure** if $\nabla^{0,1} = \bar\partial$.

DEFINITION: An Hermitian holomorphic vector bundle is a smooth complex vector bundle equipped with a Hermitian metric and a holomorphic structure operator $\bar\partial$.

DEFINITION: A Chern connection on a holomorphic Hermitian vector bundle is a connection compatible with the holomorphic structure and preserving the metric.

THEOREM: On any holomorphic Hermitian vector bundle, **the Chern connection exists, and is unique.**

REMINDER: Curvature of a connection

DEFINITION: Let $\nabla: B \longrightarrow B \otimes \Lambda^1 M$ be a connection on a smooth budnle. Extend it to an operator on B-valued forms

$$B \xrightarrow{\nabla} \Lambda^1(M) \otimes B \xrightarrow{\nabla} \Lambda^2(M) \otimes B \xrightarrow{\nabla} \Lambda^3(M) \otimes B \xrightarrow{\nabla} ...$$

using $\nabla(\eta \otimes b) = d\eta + (-1)^{\tilde{\eta}}\eta \wedge \nabla b$. The operator $\nabla^2: B \longrightarrow B \otimes \Lambda^2(M)$ is called **the curvature** of ∇.

REMARK: The algebra of $\operatorname{End}(B)$-valued forms naturally acts on $\Lambda^* M \otimes B$. The curvature satisfies $\nabla^2(fb) = d^2 fb + df \wedge \nabla b - df \wedge \nabla b + f\nabla^2 b = f\nabla^2 b$, hence it is $C^\infty M$-linear. **We consider it as an $\operatorname{End}(B)$-valued 2-form on M.**

PROPOSITION: (Bianchi identity) Clearly, $[\nabla, \nabla^2] = [\nabla^2, \nabla] + [\nabla, \nabla^2] = 0$, hence $[\nabla, \nabla^2] = 0$. This gives **Bianchi identity:** $\nabla(\Theta_B) = 0$, where Θ is considered as a section of $\Lambda^2(M) \otimes \operatorname{End}(B)$, and $\nabla: \Lambda^2(M) \otimes \operatorname{End}(B) \longrightarrow \Lambda^3(M) \otimes \operatorname{End}(B)$. the operator defined above

REMINDER: Curvature of a holomorphic line bundle

REMARK: If B is a line bundle, $\operatorname{End} B$ is trivial, and **the curvature Θ_B of B is a closed 2-form.**

DEFINITION: Let ∇ be a unitary connection in a line bundle. The cohomology class $c_1(B) := \frac{\sqrt{-1}}{2\pi}[\Theta_B] \in H^2(M)$ is called **the real first Chern class of a line bunlde B**.

An exercise: Check that $c_1(B)$ is independent from a choice of ∇.

REMARK: When speaking of a **"curvature of a holomorphic bundle"**, one usually means the curvature of a Chern connection.

REMARK: Let B be a holomorphic Hermitian line bundle, and b its non-degenerate holomorphic section. Denote by η a (1,0)-form which satisfies $\nabla^{1,0}b = \eta \otimes b$. Then $d|b|^2 = \operatorname{Re} g(\nabla^{1,0}b, b) = \operatorname{Re} \eta |b|^2$. **This gives $\nabla^{1,0}b = \frac{\partial |b|^2}{|b|^2} b = 2\partial \log |b| b$.**

REMARK: Then $\Theta_B(b) = 2\bar\partial\partial \log |b| b$, **that is, $\Theta_B = -2\partial\bar\partial \log |b|$.**

COROLLARY: If $g' = e^{2f}g$ – two metrics on a holomorphic line bundle, Θ, Θ' their curvatures, **one has $\Theta' - \Theta = -2\partial\bar\partial f$**

$\partial\overline{\partial}$-lemma

THEOREM: (“$\partial\overline{\partial}$-lemma”)
Let M be a compact Kaehler manifold, and $\eta\Lambda^{p,q}(M)$ an exact form. Then $\eta = \partial\overline{\partial}\alpha$, for some $\alpha \in \Lambda^{p-1,q-1}(M)$.

Its proof uses Hodge theory.

COROLLARY: Let (L, h) be a holomorphic line bundle on a compact complex manifold, Θ its curvature, and η a (1,1)-form in the same cohomology class as $[\Theta]$. **Then there exists a Hermitian metric h' on L such that its curvature is equal to η.**

Proof: Let Θ' be the curvature of the Chern connection associated with h'. Then $\Theta' - \Theta = -2\partial\overline{\partial}f$, wgere $f = \log(h'h^{-1})$. Then $\Theta' - \Theta = \eta - \Theta = -2\partial\overline{\partial}f$ **has a solution f by $\partial\overline{\partial}$-lemma,** because $\eta - \Theta$ is exact. ∎

Calabi-Yau manifolds

REMARK: Let B be a line bundle on a manifold. Using the long exact sequence of cohomology associated with the exponential sequence

$$0 \longrightarrow \mathbb{Z}_M \longrightarrow C^\infty M \longrightarrow (C^\infty M)^* \longrightarrow 0,$$

we obtain $0 \longrightarrow H^1(M, (C^\infty M)^*) \longrightarrow H^2(M, \mathbb{Z}) \longrightarrow 0$**.**

DEFINITION: Let B be a complex line bundle, and ξ_B its defining element in $H^1(M, (C^\infty M)^*)$. Its image in $H^2(M, \mathbb{Z})$ is called **the integer first Chern class** of B, denoted by $c_1(B, \mathbb{Z})$ or $c_1(B)$.

REMARK: A complex line bundle B is (topologically) trivial if and only if $c_1(B, \mathbb{Z}) = 0$.

THEOREM: (Gauss-Bonnet) A real Chern class of a vector bundle **is an image of the integer Chern class $c_1(B, \mathbb{Z})$** under the natural homomorphism $H^2(M, \mathbb{Z}) \longrightarrow H^2(M, \mathbb{R})$.

DEFINITION: A first Chern class of a complex n-manifold is $c_1(\Lambda^{n,0}(M))$.

DEFINITION:
A Calabi-Yau manifold is a compact Kaehler manifold with $c_1(M, \mathbb{Z}) = 0$.

Ricci form of a Kähler manifold

THEOREM: (Bogomolov) Let M be a compact Kähler n-manifold with $c_1(M,\mathbb{Z}) = 0$. **Then the canonical bundle $K_M := \Omega^n(M)$ is trivial.**

Proof: Follows from the Calabi-Yau theorem (later today). ■

In other words, a manifold is Calabi-Yau if and only if its canonical bundle is trivial.

DEFINITION: Let (M,ω) be a Kähler manifold. The metric on K_M can be written as $|\Omega|^2 = \frac{\Omega\wedge\overline{\Omega}}{\omega^n}$. The **Ricci form** on M is the curvature of the Chern connection on K_M. The manifold M is **Ricci-flat** if its Ricci form vanishes.

REMARK: Since a canonical bundle K_M of a Calabi-Yau manifold is trivial, it admits a metric with trivial connection. Calabi conjectured that **this metric on K_M is induced by a Kähler metric ω on M** and proved that such a metric is unique for any cohomology class $[\omega] \in H^{1,1}(M,\mathbb{R})$. Yau proved that it always exists.

DEFINITION: A Ricci-flat Kähler metric is called **Calabi-Yau metric**.

Calabi-Yau theorem and Monge-Ampère equation

REMARK: Let (M,ω) be a Kähler n-fold, and Ω a non-degenerate section of $K(M)$, Then $|\Omega|^2 = \frac{\Omega\wedge\overline{\Omega}}{\omega^n}$. If ω_1 is a new Kaehler metric on (M,I), h, h_1 the associated metrics on $K(M)$, then $\frac{h}{h_1} = \frac{\omega_1^n}{\omega^n}$.

REMARK: For two metrics ω_1, ω in the same Kähler class, one has $\omega_1 - \omega = dd^c\varphi$**, for some function** φ (dd^c-lemma).

COROLLARY: A metric $\omega_1 = \omega + \partial\overline{\partial}\varphi$ **is Ricci-flat if and only if** $(\omega + dd^c\varphi)^n = \omega^n e^f$, where $-2\partial\overline{\partial} f = \Theta_{K,\omega}$ **(such f exists by $\partial\overline{\partial}$-lemma).**

Proof. Step 1: For such f, φ, one has $\log\frac{h}{h_1} = -\log e^f = -f$. As shown above, the corresponding curvatures are related as $\Theta_{K,\omega_1} - \Theta_{K,\omega} = -2\partial\overline{\partial}\log(h/h_1)$. This gives

$$\Theta_{K,\omega_1} = \Theta_{K,\omega} - 2\partial\overline{\partial}\log(h/h_1) = \Theta_{K,\omega} - 2\partial\overline{\partial} f.$$

Proof. Step 2: Therefore, ω_1 is Ricci-flat if and only if $\Theta_{K,\omega} - 2\partial\overline{\partial} f$. ■

To find a Ricci-flat metric **it remains to solve an equation** $(\omega + dd^c\varphi)^n = \omega^n e^f$ **for a given f.**

The complex Monge-Ampère equation

To find a Ricci-flat metric **it remains to solve an equation $(\omega + dd^c\varphi)^n = \omega^n e^f$ for a given f.**

THEOREM: (Calabi-Yau) Let (M, ω) be a compact Kaehler n-manifold, and f any smooth function. **Then there exists a unique up to a constant function φ** such that $(\omega + \sqrt{-1}\,\partial\bar\partial\varphi)^n = Ae^f\omega^n$, where A is a positive constant obtained from the formula $\int_M Ae^f\omega^n = \int_M \omega^n$.

DEFINITION:

$$(\omega + \sqrt{-1}\,\partial\bar\partial\varphi)^n = Ae^f\omega^n,$$

is called **the Monge-Ampere equation.**

Uniqueness of solutions of complex Monge-Ampere equation

PROPOSITION: (Calabi) **A complex Monge-Ampere equation has at most one solution,** up to a constant.

Proof. Step 1: Let ω_1, ω_2 be solutions of Monge-Ampere equation. Then $\omega_1^n = \omega_2^n$. By construction, one has $\omega_2 = \omega_1 + \sqrt{-1}\,\partial\bar\partial\psi$. **We need to show $\psi = const$.**

Step 2: $\omega_2 = \omega_1 + \sqrt{-1}\,\partial\bar\partial\psi$ gives

$$0 = (\omega_1 + \sqrt{-1}\,\partial\bar\partial\psi)^n - \omega_1^n = \sqrt{-1}\,\partial\bar\partial\psi \wedge \sum_{i=0}^{n-1} \omega_1^i \wedge \omega_2^{n-1-i}.$$

Step 3: Let $P := \sum_{i=0}^{n-1} \omega_1^i \wedge \omega_2^{n-1-i}$. This is a positive $(n-1, n-1)$-form. **There exists a Hermitian form ω_3 on M such that $\omega_3^{n-1} = P$.**

Step 4: Since $\sqrt{-1}\,\partial\bar\partial\psi \wedge P = 0$, this gives $\psi\partial\bar\partial\psi \wedge P = 0$. Stokes' formula implies

$$0 = \int_M \psi \wedge \partial\bar\partial\psi \wedge P = -\int_M \partial\psi \wedge \bar\partial\psi \wedge P = -\int_M |\partial\psi|_3^2 \omega_3^n.$$

where $|\cdot|_3$ is the metric associated to ω_3. **Therefore $\bar\partial\psi = 0$.** ■

Levi-Civita connection and Kähler geometry

DEFINITION: Let (M, g) be a Riemannian manifold. A connection ∇ is called **orthogonal** if $\nabla(g) = 0$. It is called **Levi-Civita** if it is torsion-free.

THEOREM: ("the main theorem of differential geometry") **For any Riemannian manifold, the Levi-Civita connection exists, and it is unique**.

THEOREM: Let (M, I, g) be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) (M, I, g) **is Kähler**

(ii) One has $\nabla(I) = 0$, where ∇ is the Levi-Civita connection.

Holonomy group

DEFINITION: (Cartan, 1923) Let (B,∇) be a vector bundle with connection over M. For each loop γ based in $x \in M$, let $V_{\gamma,\nabla}: B|_x \longrightarrow B|_x$ be the corresponding parallel transport along the connection. The **holonomy group** of (B,∇) is a group generated by $V_{\gamma,\nabla}$, for all loops γ. If one takes all contractible loops instead, $V_{\gamma,\nabla}$ generates **the local holonomy**, or **the restricted holonomy** group.

REMARK: A bundle is **flat** (has vanishing curvature) **if and only if its restricted holonomy vanishes.**

REMARK: If $\nabla(\varphi)=0$ for some tensor $\varphi \in B^{\otimes i}\otimes (B^*)^{\otimes j}$, **the holonomy group preserves φ.**

DEFINITION: **Holonomy of a Riemannian manifold** is holonomy of its Levi-Civita connection.

EXAMPLE: Holonomy of a Riemannian manifold lies in $O(T_xM, g|_x) = O(n)$.

EXAMPLE: Holonomy of a Kähler manifold lies in $U(T_xM, g|_x, I|_x) = U(n)$.

REMARK: The holonomy group **does not depend on the choice of a point $x \in M$.**

The Berger's list

THEOREM: (de Rham) A complete, simply connected Riemannian manifold with non-irreducible holonomy **splits as a Riemannian product.**

THEOREM: (Berger's theorem, 1955) Let G be an irreducible holonomy group of a Riemannian manifold which is not locally symmetric. **Then G belongs to the Berger's list:**

Berger's list	
Holonomy	*Geometry*
$SO(n)$ acting on $\mathbb{R}^n$	Riemannian manifolds
$U(n)$ acting on $\mathbb{R}^{2n}$	Kähler manifolds
$SU(n)$ acting on $\mathbb{R}^{2n}$, $n > 2$	Calabi-Yau manifolds
$Sp(n)$ acting on $\mathbb{R}^{4n}$	hyperkähler manifolds
$Sp(n) \times Sp(1)/\{\pm 1\}$ acting on $\mathbb{R}^{4n}$, $n > 1$	quaternionic-Kähler manifolds
G_2 acting on $\mathbb{R}^7$	G_2-manifolds
$Spin(7)$ acting on $\mathbb{R}^8$	$Spin(7)$-manifolds

Chern connection

DEFINITION: Let B be a holomorphic vector bundle on a complex manifold, and $\overline{\partial}: B_{C^\infty} \longrightarrow B_{C^\infty} \otimes \Lambda^{0,1}(M)$ an operator mapping $b \otimes f$ to $b \otimes \overline{\partial} f$, where $b \in B$ is a holomorphic section, and f a smooth function. This operator is called **a holomorphic structure operator** on B. **It is correctly defined, because $\overline{\partial}$ is $\mathcal{O}_M$-linear.**

REMARK: A section $b \in B$ is holomorphic iff $\overline{\partial}(b) = 0$

DEFINITION: Let (B, ∇) be a smooth bundle with connection and a holomorphic structure $\overline{\partial}: B \longrightarrow \Lambda^{0,1}(M) \otimes B$. Consider the Hodge decomposition of ∇, $\nabla = \nabla^{0,1} + \nabla^{1,0}$. We say that ∇ is **compatible with the holomorphic structure** if $\nabla^{0,1} = \overline{\partial}$.

DEFINITION: An Hermitian holomorphic vector bundle is a complex vector bundle equipped with a Hermitian metric and a holomorphic structure.

DEFINITION: A Chern connection on a holomorphic Hermitian vector bundle is a connection compatible with the holomorphic structure and preserving the metric.

THEOREM: On any holomorphic Hermitian vector bundle, **the Chern connection exists, and is unique.**

Calabi-Yau manifolds

DEFINITION:
A Calabi-Yau manifold is a compact Kaehler manifold with $c_1(M,\mathbb{Z})=0$.

DEFINITION: Let (M,I,ω) be a Kaehler n-manifold, and $K(M):=\Lambda^{n,0}(M)$ its **canonical bundle.** We consider $K(M)$ as a holomorphic line bundle, $K(M)=\Omega^n M$. The natural Hermitian metric on $K(M)$ is written as

$$(\alpha,\alpha')\longrightarrow \frac{\alpha\wedge\overline{\alpha}'}{\omega^n}.$$

Denote by Θ_K the curvature of the Chern connection on $K(M)$. The **Ricci curvature** Ric of M is a symmetric 2-form $\operatorname{Ric}(x,y)=\Theta_K(x,Iy)$.

DEFINITION: A Kähler manifold is called **Ricci-flat** if its Ricci curvature vanishes.

THEOREM: (Calabi-Yau)
Let (M,I,g) be Calabi-Yau manifold. **Then there exists a unique Ricci-flat Kaehler metric in any given Kaehler class.**

REMARK: Converse is also true: **any Ricci-flat Kähler manifold has a finite covering which is Calabi-Yau.** This is due to Bogomolov.

Bochner's vanishing

THEOREM: (Bochner vanishing theorem) On a compact Ricci-flat Calabi-Yau manifold, **any holomorphic p-form η is parallel** with respect to the Levi-Civita connection: $\nabla(\eta) = 0$.

REMARK: Its proof is based on spinors: η gives a harmonic spinor, and **on a Ricci-flat Riemannian spin manifold, any harmonic spinor is parallel.**

DEFINITION: A **holomorphic symplectic manifold** is a manifold admitting a non-degenerate, holomorphic symplectic form.

REMARK: A holomorphic symplectic manifold is Calabi-Yau. The top exterior power of a holomorphic symplectic form **is a non-degenerate section of canonical bundle.**

Hyperkähler manifold

REMARK: Due to Bochner's vanishing, **holonomy of Ricci-flat Calabi-Yau manifold lies in $SU(n)$**, and **holonomy of Ricci-flat holomorphically symplectic manifold lies in $Sp(n)$** (a group of complex unitary matrices preserving a complex-linear symplectic form).

DEFINITION: A holomorphically symplectic Kähler manifold with holonomy in $Sp(n)$ is called **hyperkähler**.

REMARK: Since $Sp(n) = SU(\mathbb{H}, n)$, a **hyperkähler manifold admits quaternionic action in its tangent bundle.**

EXAMPLES.

EXAMPLE: An even-dimensional complex vector space.

EXAMPLE: An even-dimensional complex torus.

EXAMPLE: A non-compact example: $T^*\mathbb{C}P^n$ (Calabi).

REMARK: $T^*\mathbb{C}P^1$ **is a resolution of a singularity** $\mathbb{C}^2/\pm 1$**.**

REMARK: Let M be a 2-dimensional complex manifold with holomorphic symplectic form outside of singularities, which are all of form $\mathbb{C}^2/\pm 1$. Then **its resolution is also holomorphically symplectic.**

EXAMPLE: Take a 2-dimensional complex torus T, then all the singularities of $T/\pm 1$ are of this form. Its resolution $\widetilde{T/\pm 1}$ is called **a Kummer surface**. **It is holomorphically symplectic**.

REMARK: Take a symmetric square $\operatorname{Sym}^2 T$, with a natural action of T, and let $T^{[2]}$ be a blow-up of a singular divisor. **Then** $T^{[2]}$ **is naturally isomorphic to the Kummer surface** $\widetilde{T/\pm 1}$**.**

K3 surfaces

DEFINITION: A K3-surface is a deformation of a Kummer surface.

"K3: Kummer, Kähler, Kodaira" (a name is due to A. Weil).

"Faichan Kangri (K3) is the 12th highest mountain on Earth."

THEOREM: Any complex compact surface with $c_1(M) = 1$ and $H^1(M) = 0$ **is isomorphic to K3.** Moreover, **it is hyperkähler.**

Hilbert schemes

REMARK: **A complex surface** is a 2-dimensional complex manifold.

DEFINITION: A **Hilbert scheme** $M^{[n]}$ of a complex surface M is a classifying space of all ideal sheaves $I \subset \mathcal{O}_M$ for which the quotient $\mathcal{O}_M/I$ has dimension n over $\mathbb{C}$.

REMARK: A Hilbert scheme **is obtained as a resolution of singularities** of the symmetric power $\operatorname{Sym}^n M$.

THEOREM: (Fujiki, Beauville) **A Hilbert scheme of a hyperkähler surface is hyperkähler.**

EXAMPLE: **A Hilbert scheme of K3**.

EXAMPLE: Let T is a torus. Then it acts on its Hilbert scheme freely and properly by translations. For $n=2$, the quotient $T^{[n]}/T$ is a Kummer K3-surface. For $n>2$, it is called **a generalized Kummer variety**.

REMARK: There are 2 more "sporadic" examples of compact hyperkähler manifolds, constructed by K. O'Grady. **All known compact hyperkaehler manifolds are these 2 and the three series:** tori, Hilbert schemes of K3, and generalized Kummer.

Bogomolov's decomposition theorem

THEOREM: (Cheeger-Gromoll) Let M be a compact Ricci-flat Riemannian manifold with $\pi_1(M)$ infinite. **Then a universal covering of M is a product of $\mathbb{R}$ and a Ricci-flat manifold.**

COROLLARY: A fundamental group of a compact Ricci-flat Riemannian manifold is **"virtually polycyclic": it is projected to a free abelian subgroup with finite kernel.**

REMARK: This is equivalent to any compact Ricci-flat manifold having a finite covering which has free abelian fundamental group.

REMARK: This statement contains the Bieberbach's solution of Hilbert's 18-th problem on classification of crystallographic groups.

THEOREM: (Bogomolov's decomposition) Let M be a compact, Ricci-flat Kaehler manifold. **Then there exists a finite covering $\tilde{M}$ of M which is a product of Kaehler manifolds of the following form:**

$$\tilde{M} = T \times M_1 \times ... \times M_i \times K_1 \times ... \times K_j,$$

with all M_i, K_i simply connected, T a torus, and $\mathcal{H}\text{ol}(M_l) = Sp(n_l)$, $\mathcal{H}\text{ol}(K_l) = SU(m_l)$

Harmonic forms

Let V be a vector space. **A metric g on V induces a natural metric on each of its tensor spaces:** $g(x_1 \otimes x_2 \otimes ... \otimes x_k, x'_1 \otimes x'_2 \otimes ... \otimes x'_k) = g(x_1, x'_1)g(x_2, x'_2)...g(x_k, x'_k)$.

This gives a natural positive definite scalar product on differential forms over a Riemannian manifold (M, g): $g(\alpha, \beta) := \int_M g(\alpha, \beta) \operatorname{Vol}_M$. The topology induced by this metric is called **L^2-topology.**

DEFINITION: Let d be the de Rham differential and d^* denote the adjoint operator. The **Laplace operator** is defined as $\Delta := dd^* + d^*d$. A form is called **harmonic** if it lies in $\ker \Delta$.

THEOREM: The image of Δ is closed in L^2-topology on differential forms.

REMARK: This is a very difficult theorem!

REMARK: On a compact manifold, the form η is **harmonic iff $d\eta = d^*\eta = 0$.** Indeed, $(\Delta x, x) = (dx, dx) + (d^*x, d^*x)$.

COROLLARY: This defines a map $\mathcal{H}^i(M) \xrightarrow{\tau} H^i(M)$ from harmonic forms to cohomology.

Hodge theory

THEOREM: (Hodge theory for Riemannian manifolds)
On a compact Riemannian manifold, the map $\mathcal{H}^i(M) \xrightarrow{\tau} H^i(M)$ to cohomology **is an isomorphism.**

Proof. Step 1: $\ker d \perp \operatorname{im} d^*$ and $\operatorname{im} d \perp \ker d^*$. Therefore, **a harmonic form is orthogonal to** $\operatorname{im} d$**.** This implies that τ **is injective**.

Step 2: $\eta \perp \operatorname{im} \Delta$ **if and only if** η **is harmonic.** Indeed, $(\eta, \Delta x) = (\Delta x, x)$.

Step 3: Since $\operatorname{im} \Delta$ is closed, **every closed form** η **is decomposed as** $\eta = \eta_h + \eta'$**,** where η_h is harmonic, and $\eta' = \Delta\alpha$.

Step 4: When η is closed, η' is also closed. Then $0 = (d\eta, d\alpha) = (\eta, d^* d\alpha) = (\Delta\alpha, d^* d\alpha) = (dd^*\alpha, d^* d\alpha) + (d^* d\alpha, d^* d\alpha)$. The term $(dd^*\alpha, d^* d\alpha)$ vanishes, because $d^2 = 0$, hence $(d^* d\alpha, d^* d\alpha) = 0$. This gives $d^* d\alpha = 0$, and $(d^* d\alpha, \alpha) = (d\alpha, d\alpha) = 0$. We have shown that **for any closed** η **decomposing as** $\eta = \eta_h + \eta'$**, with** $\eta' = \Delta\alpha$**,** α **is closed**

Step 5: This gives $\eta' = dd^*\alpha$, hence η **is a sum of an exact form and a harmonic form.** ■

REMARK: This gives a way of obtaining the Poincare duality via PDE.

Hodge decomposition on cohomology

THEOREM: *(this theorem will be proven in the next lecture)*
On a compact Kaehler manifold M, **the Hodge decomposition is compatible with the Laplace operator.** This gives a decomposition of cohomology, $H^i(M) = \bigoplus_{p+q=i} H^{p,q}(M)$, with $\overline{H^{p,q}(M)} = H^{q,p}(M)$.

COROLLARY: $H^p(M)$ **is even-dimensional for odd** p**.**

The Hodge diamond:

$$
\begin{array}{ccccccc}
 & & & H^{n,n} & & & \\
 & & H^{n,n-1} & & H^{n-1,n} & & \\
 & H^{n,n-2} & & H^{n-1,n-1} & & H^{n-2,n} & \\
\vdots & \vdots & & \vdots & & \vdots & \vdots \\
 & H^{2,0} & & H^{1,1} & & H^{0,2} & \\
 & & H^{1,0} & & H^{0,1} & & \\
 & & & H^{0,0} & & &
\end{array}
$$

REMARK: $H^{p,0}(M)$ **is the space of holomorphic** p**-forms.** Indeed, $dd^* + d^*d = 2(\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})$, hence **a holomorphic form on a compact Kähler manifold is closed.**

Holomorphic Euler characteristic

DEFINITION: A holomorphic Euler characteristic $\chi(M)$ of a Kähler manifold is a sum $\sum(-1)^p \dim H^{p,0}(M)$.

THEOREM: (Riemann-Roch-Hirzebruch) For an n-fold, $\chi(M)$ **can be expressed as a polynomial expressions of the Chern classes,** $\chi(M) = td_n$ where td_n is an n-th component of the Todd polynomial,

$$td(M) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + c_1c_3 + 3c_2^2 2 - c_4) + ...$$

REMARK: The Chern classes are obtained as polynomial expression of the curvature (Gauss-Bonnet). Therefore $\chi(\tilde{M}) = p\chi(M)$ **for any unramified** p**-fold covering** $\tilde{M} \longrightarrow M$**.**

REMARK: Bochner's vanishing and the classical invariants theory imply:

1. When $\mathcal{H}\text{ol}(M) = SU(n)$, we have $\dim H^{p,0}(M) = 1$ **for** $p = 1, n$**, and 0 otherwise.** In this case, $\chi(M) = 2$ for even n and 0 for odd.

2. When $\mathcal{H}\text{ol}(M) = Sp(n)$,we have $\dim H^{p,0}(M) = 1$ **for even** p $0 \leqslant p \leqslant 2n$**, and 0 otherwise.** In this case, $\chi(M) = n + 1$.

COROLLARY: $\pi_1(M) = 0$ if $\mathcal{H}\text{ol}(M) = Sp(n)$, or $\mathcal{H}\text{ol}(M) = SU(2n)$. If $\mathcal{H}\text{ol}(M) = SU(2n+1)$, $\pi_1(M)$ is finite.