Complex geometry: exam

Handouts score is given by the formula $t = 10 + 5a + 8b$, where a is the number of non-completed handouts with at least 1/2 of exercises credited, and b the number of completed handouts.

Each student receives a random selection of test problems (the output of the randomizer is printed on a separate sheet). The final score for the course is $s = p + [t/10]$, where p is the total number of points for the exam. The exam is oral.

1 Almost complex manifolds

Exercise 1.1 (2 points). Let ω be a non-degenerate 2-form on a Riemannian manifold, and ∇ its Levi-Civita connection. Assume that $\nabla(\omega) = 0$. Prove that M admits a complex structure I such that $\nabla(I) = 0$.

Exercise 1.2. Let f be a holomorphic function on an almost complex manifold. Suppose that $|f|$ is constant. Prove that f is constant.

Exercise 1.3. Construct a G-invariant complex structure on G/H, or prove it does not exist.

a. (3 points) $G = SL(6)$, $H = SO(6)$

b. (2 points) $G = GL(8)$, $H = SO(8)$

Exercise 1.4 (2 points). Let (M, I) be an almost complex manifold, $\dim_{\mathbb{C}} M = n$, $U \subset M$ a dense, open subset, and $\Omega \in \Lambda^{n,0}(U)$ a non-degenerate $(n, 0)$-form. Assume that $d\Omega = 0$. Prove that the almost complex structure I is integrable.

Exercise 1.5. Let (M, I) be an almost complex manifold, and $d = d^{-1,2} + d^{0,1} + d^{0,1} + d^{2,-1}$ the Hodge decomposition of its de Rham differential.

a. Prove that $\{d^{1,0}, d^{1,0}\} = 0$, or find a counterexample.

b. Prove that $[d^{2,-1}, \{d^{1,0}, d^{0,1}\}] = 0$.

Exercise 1.6. Let (M, I) be an almost complex manifold, and W the Weil operator, acting on (p, q)-forms as $W(\eta) = \sqrt{-1}(p-q)\eta$. Prove that I is integrable if and only if $I^{-1}dI - [W, d] = 0$.

Definition 1.1. Holomorphic differential on an almost complex manifold is a closed $(1, 0)$-form.

Exercise 1.7. Let M be an almost complex manifold, and $\phi : M \longrightarrow \mathbb{R}$ a function which satisfies $dId(\phi) = 0$. Prove that M admits a non-zero holomorphic differential.

2 Symplectic, complex and Kähler structures

Exercise 2.1. Let ω be a non-degenerate 2-form on a Riemannian manifold, and ∇ its Levi-Civita connection. Assume that $\nabla(\omega) = 0$. Prove that M admits a complex structure I such that $\nabla(I) = 0$.

Exercise 2.2. Construct a G-invariant Hermitian structure on G/H and prove that it is Kähler.

a. (2 points) $G = SO(2n)$, $H = U(n)$.

b. (2 points) $G = U(p, q)$, $H = U(p) \times U(q)$.

c. (2 points) $G = U(p + q)$, $H = U(p) \times U(q)$.

Exercise 2.3 (2 points). Let G be a compact, connected Lie group with a left invariant complex structure and a left invariant Kähler metric. Prove that G is commutative.

Definition 2.1. Let (M, I) be a complex manifold, and $X \in TM$ a real vector field. Recall that X is called **holomorphic** if the corresponding diffeomorphisms are holomorphic, or, equivalently, $\operatorname{Lie}_X I = 0$.

Exercise 2.4 (3 points). Let (M, I) be a complex manifold, X a vector field, and ∇ a torsion-free connection which satisfies $\nabla(I) = 0$. Prove that X is holomorphic if and only if $\nabla X \in \Lambda^1(M) \otimes TM$, considered as an endomorphism of TM, is complex linear.

Exercise 2.5. Let (M, I, ω) be an almost complex Hermitian manifold, with $d\omega = 0$. Find the dimension of the Lie superalgebra generated by L, Λ, d, where $L(\eta) = \omega \wedge \eta$, and $\Lambda = *L*$.

Definition 2.2. Holomorphic form on a complex manifold is a $(p, 0)$-form η which satisfies $\bar\partial\eta = 0$.

Exercise 2.6. Let Ω be a holomorphic $(n-1)$-form on a compact complex manifold M with $\dim_{\mathbb{C}} M = n$. Prove that $d\Omega = 0$.

Exercise 2.7. Let η be a real non-zero (1,1)-form on a compact Kähler manifold (M, ω), $\dim_{\mathbb{C}} M = 2$. Assume that $\omega \wedge \eta = 0$. Prove that $\int_M \eta \wedge \eta < 0$

3 Hodge theory and its applications

Exercise 3.1. Let η be a holomorphic form on a Kähler manifold. Prove that η is harmonic.

Exercise 3.2. Let ω be a non-degenerate 2-form on a $2n$-dimensional smooth manifold, and $d(\omega^k) = 0$ for some k satisfying $0 < k < n-1$. Prove that $d\omega = 0$.

Exercise 3.3 (2 points). Let M be a closed ball in $\mathbb{R}^n$ with a Riemannian metric g which smoothly extends to its boundary, and $\alpha \in \Lambda^k(M)$ a differential form, also smoothly extending to its boundary. Prove that $\alpha \in \operatorname{im} \Delta$, where Δ is the Laplace operator associated with g.

Exercise 3.4. Let F be an exact holomorphic n-form on an n-dimensional compact complex manifold. Prove that $F = 0$.

Exercise 3.5 (2 points). Let M be a compact complex manifold, $\dim_{\mathbb{C}} M = 2$. Prove that all holomorphic forms on M are closed.

Exercise 3.6. Let θ be a closed holomorphic 1-form on a simply connected compact complex manifold (not necessarily Kähler). Prove that $\theta = 0$.

Exercise 3.7. Let η be a (1,1)-form with compact support on $M \cong \mathbb{C}$. Prove that there exists $f \in C^\infty M$ with compact support such that $\eta = dd^c f$, or find a counterexample.

Exercise 3.8 (2 points). Let M be a compact Riemannian manifold, $\mathcal{H}^i$ the sheaf of harmonic i-forms, and $\nu : \mathcal{H}^i \longrightarrow \Lambda^i(M)$ the tautological embedding. Prove that the sequence

$$0 \longrightarrow \mathcal{H}^i \xrightarrow{\nu} \Lambda^i(M) \xrightarrow{\Delta} \Lambda^i(M) \longrightarrow 0$$

is exact.

4 Geometry and topology of Kähler manifolds

Exercise 4.1. Let $M = CP^4 \times \mathbb{C}P^4 \times \mathbb{C}P^4$. Prove that M does not admit a Kähler structure with non-standard orientation.

Exercise 4.2. Let M be a compact Kähler manifold, $\dim_{\mathbb{C}} M = 4$. Prove that M does not admit a Kähler structure with opposite orientation or find a counterexample.

Exercise 4.3 (2 points). Let M be a compact complex manifold, and $\pi_1(M) \cong \Gamma$ where Γ is a group of upper triangular integer matrices 4x4 with 1 on diagonal. Prove that M does not admit a Kähler structure.

Exercise 4.4. For any given $n > 2$ find a $2n$-dimensional connected simply connected manifold with $b_{2i} \neq 0$, $i = 0, 1, ..., n$ not admitting a symplectic structure.

Exercise 4.5. Let $\eta \in \Lambda^{1,1}(M)$ be a closed form on a compact Kähler manifold, $\dim_{\mathbb{C}} M = 2$. Assume that $\eta \wedge \omega = 0$. Prove that η is harmonic.

Exercise 4.6. Let M be a compact Kähler manifold. **Kähler cone** of M is the set $K(M) \subset H^{1,1}_{\mathbb{R}}(M)$ of all cohomology classes of Kähler forms, where $H^{1,1}_{\mathbb{R}}(M) = H^{1,1}(M) \cap H^2(M, \mathbb{R})$. Prove that $K(M)$ is open in $H^{1,1}_{\mathbb{R}}(M)$.

Exercise 4.7 (2 points). Let $T = \mathbb{C}^2/\mathbb{Z}^4$ be a complex torus. Prove that $\mathsf{rk}\left[H^{1,1}(M) \cap H^2(M, \mathbb{Z})\right]$ is always positive, or find a counterexample.

5 Group action on manifolds

Exercise 5.1. Let V be a representation of a finite group G (not necessarily finitely-dimensional). Prove that every vector $v \in V$ admits a finite decomposition $v = \sum v_i$ where each v_i belongs to a finitely-dimensional representation of G.

Exercise 5.2. Bi-invariant forms on Lie groups are forms which are invariant under the left and right group action. Let G be a compact Lie group equipped with a bi-invariant metric.

a. (2 points) Prove that all bi-invariant differential forms on G are harmonic.

b. Prove that all harmonic forms are bi-invariant.

Exercise 5.3. Let G be a finite group freely acting on a compact manifold M. Prove that $H^*(M/G) = H^*(M)^G$, where $H^*(M)^G$ denotes invariants of the action of G on cohomology.

Exercise 5.4. Let M be a complex manifold, admtting a holomorphic vector field with isolated fixed points only. Prove the the topological Euler characteristic of M is non-negative.

Exercise 5.5. Let (M, g, ω) be a compact Kähler manifold, ∇ the Levi-Civita connection, and $X \in TM$ a vector field which satisfies $\nabla(X) = 0$.

a. (2 points) Prove that X is Killing, that is, the corresponding diffeomorphism flow acts by isometries.

b. (3 points) Prove that X is holomorphic.

Exercise 5.6. Let M be a compact Riemannian manifold, and G a group acting on M by isometries. Denote by G_0 the image of G in $\operatorname{Aut}(H^*(M))$. Prove that G_0 is finite.

Exercise 5.7. Let G be a finite group acting on $M = \mathbb{C}^n$ by holomorphic maps. Prove that M admits a G-invariant holomorphic differential.