Complex geometry handout 1: Complex structure operators

Exercise 1.1. Prove that the group $GL(n, \mathbb{C})$ of complex automorphisms of $\mathbb{C}^n$ is connected.

Definition 1.1. Let V be a vector space. A **complex structure** on V is an operator $I \in \operatorname{End}(V)$ which satisfies $I^2 = -\operatorname{\mathsf{Id}}$.

Exercise 1.2. Let V be a real 4-dimensional vector space, equipped with a Euclidean metric, and S the space of all orthogonal complex structures $I \in \operatorname{End}(V)$. Prove that S is a disconnected union of two 2-dimensional spheres.

Exercise 1.3. Let V be a real 4-dimensional vector space, equipped with a scalar product g of signature $(3, 1)$. Prove that the space $\Lambda^2(V)$ of anti-symmetric 2-forms is equipped with an $SO(3, 1)$-invariant complex structure.

Exercise 1.4. Let V be a 4-dimensional real vector space, and U the space of antisymmetric 2-forms Ω on $V \otimes_{\mathbb{R}} \mathbb{C}$ such that $\Omega \wedge \Omega = 0$ and $\Omega \wedge \bar{\Omega} \neq 0$. Denote by $\mathbb{P}U$ the projectivization of U, that is, the quotient $U/\mathbb{C}^*$, under the standard $\mathbb{C}^*$-action. Construct a $GL(V)$-invariant bijective correspondence between $\mathbb{P}U$ and the space of complex structures on V.

Definition 1.2. Let M be a manifold. An endomorphism $I \in \operatorname{End}(TM)$, $I^2 = -\operatorname{\mathsf{Id}}_{TM}$ is called **an almost complex structure**. An I-invariant Riemannian form is called **Hermitian form**. A smooth map ϕ of almost complex manifold is called **holomorphic** if ifs differential commutes with I.

Definition 1.3. Let g_1, g_2 be Riemannian metrics on a smooth manifold M. They are said to be **conformal**, or **conformally equivalent** if there exists a smooth function $\lambda \in C^\infty M$ such that $g_1 = \lambda \cdot g_2$. **Conformal structure** is a metric up to conformal equivalence.

Exercise 1.5. Let I be an almost complex structure on a manifold M of real dimension 2. Prove that all Hermitian metrics on (M, I) are conformally equivalent.

Exercise 1.6. Let (M, I) be an almost complex manifold, $\dim_{\mathbb{C}} M = n$ Prove that (M, I) always admits a Hermitian metric g. Consider the orientation form ω^n, obtained as the top exterior power of the corresponding Hermitian form ω. Prove that the orientation defined by ω^n is independent from the choice of g.

Exercise 1.7. Let $f : M \longrightarrow N$ be an oriented diffeomorphism of almost complex Hermitian manifolds of real dimension 2. Prove that f is holomorphic if and only if it preserves the conformal structure.

Exercise 1.8. Prove that the space of almost complex structures on a 2-dimensional manifold is a disconnected union of two contractible sets.

Exercise 1.9. Let M be a manifold admitting a non-degenerate 2-form. Prove that M admits an almost complex structure.

Exercise 1.10. Prove that the space of almost complex structures is homotopy equivalent to the space of non-degenerate 2-forms.