Complex geometry handout 2: Distributions and almost complex structures

Exercise 2.1. Find a vector field v on a 2-dimensional torus T^2 such that all orbits of the corresponding diffeomorphism flow are dense.

Exercise 2.2. Construct a 4-manifold M and a rank 2 distribution $B \subset TM$ such that $[B, B]$ has rank 3 and $[[B, B], B]$ has rank 4.

Definition 2.1. A contact structure on an $2n+1$-dimensional manifold M is a rank $2n$ distribution $B \subset TM$ such that TM/B is a trivial rank one bundle and the Frobenius form $\Lambda^2 B \longrightarrow TM/B$ is non-degenerate.

Exercise 2.3. Let θ be a 1-form on an $2n+1$-dimensional manifold M such that $\theta \wedge (d\theta)^n$ is non-degenerate, $f \in C^\infty M$ a nowhere vanishing function, and $\theta' = f\theta$. Prove that $\theta' \wedge (d\theta')^n$ is non-degenerate.

Exercise 2.4. Let θ be a 1-form on an $2n+1$-dimensional manifold M such that $\theta \wedge (d\theta)^n$ is non-degenerate. Prove that $\ker\theta \subset TM$ is a contact distribution.

Exercise 2.5. Let M be an odd-dimensional manifold, and $B \subset TM$ a contact distribution. Prove that there exists a 1-form θ such that $B = \ker\theta$ and $\theta \wedge (d\theta)^n$ is non-degenerate.

Exercise 2.6. Construct a contact structure on a sphere S^{2n+1} for any $n = 1, 2, 3, ...$

Hint. Use the previous exercise.

Exercise 2.7. Let M be a contact manifold. Prove that M admits a pseudo-Riemannian structure of signature $(1, 2n)$.

Exercise 2.8 (*). Let M be a compact almost complex manifold, and f a holomorphic function on M. Prove that f is constant.

Exercise 2.9. Let η, η' be non-vanishing closed $(p, 0)$-forms on an almost complex manifold, satisfying $\eta = f\eta'$ for some $f \in C^\infty M$. Prove that f is holomorphic.

Definition 2.2. Let M be an almost complex manifold, and $A : \Lambda^* M \longrightarrow \Lambda^* M$ a linear map. **Hodge components** of A are operators $A^{p,q}$ such that $A = \sum_{p,q} A^{p,q}$ and $A^{p,q}(\Lambda^{i,j}(M)) \subset \Lambda^{i+p,j+q}(M)$.

Exercise 2.10. Prove that the de Rham differential on an almost complex manifold has at most 4 non-zero Hodge components: $d = d^{2,-1} + d^{1,0} + d^{0,1} + d^{-1,2}$.