Complex geometry handout 3: real analytic manifolds and integrability

In this handout, you are not allowed to invoke the Newlander-Nirenberg theorem.

Exercise 3.1. Let $\iota: M \longrightarrow M$ be a real structure on a complex manifold. Prove that ι maps complex submanifolds of M to complex submanifolds.

Exercise 3.2. Let M be a smooth manifold with a smooth action of a finite group G. Prove that the fixed point set of G is a smooth submanifold or find a counterexample.

Exercise 3.3. Let M be a smooth manifold with a smooth action of the group $\mathbb{Z}$. Prove that the fixed point set of $\mathbb{Z}$ is a smooth submanifold or find a counterexample.

Definition 3.1. Let (M, I) be an almost complex manifold. Recall that I is **integrable** if M admits a complex structure such that in local holomorphic coordinates $z_1, ..., z_n$, with I mapping dx_i to dy_i and dy_i to $-dx_i$, where x_i, y_i are real and imaginary parts of $z_i = x_i + \sqrt{-1}y_i$. An almost complex structure I is **formally integrable** if $[T^{1,0}M, T^{1,0}(M)] \subset T^{1,0}M$.

Exercise 3.4. Let (M, I) be an almost complex manifold. Assume that the bundle $\Lambda^{1,0}(M)$ is generated by differentials of holomorphic functions. Prove that the almost complex structure on M is integrable.

Exercise 3.5. Let (M, I) be an almost complex manifold, and $d = d^{2,-1} + d^{1,0} + d^{0,1} + d^{-1,2}$ the Hodge decomposition of the de Rham differential.

a. Prove that $d^{2,-1}$ and $d^{-1,2}$ are $C^\infty(M)$-linear.

b. Assume that $d^{2,-1} = d^{-1,2} = 0$. Prove that I is formally integrable.

Definition 3.2. Let G be a real Lie group, and $\mathfrak{g} = T_eG$ its Lie algebra. **The left action** of G on itself is a map $L_g(x) = gx$, defined for any $g \in G$. An almost complex structure is called **left invariant** if it is preserved by L_g for all $g \in G$.

Exercise 3.6. Let G be a real Lie group, and $\mathfrak{g} = T_eG$ its Lie algebra, and $\mathfrak{g}_\mathbb{C} := \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$ its complexification.

a. Prove that any complex structure operator on $\mathfrak{g} = T_eG$ is uniquely extended to a left-invariant almost complex structure on G.

b. Let $I \in \operatorname{End} T_e G$ be a complex structure, and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$ the Hodge decomposition of its complexification. Prove that I defines a formally integrable left-invariant complex structure on G if and only if $\mathfrak{g}^{1,0} \subset \mathfrak{g}_{\mathbb{C}}$ is a Lie subalgebra.

Exercise 3.7. Let (M, I) be an almost complex manifold.

a. Earlier, we denoted the weight (-1,2) Hodge component of de Rham differential by $d^{-1,2}$. Prove that $d^{-1,2}: \Lambda^{1,0}(M) \longrightarrow \Lambda^{0,2}(M)$ is dual to the complex conjugate of the Nijenhuis tensor $\bar{N}: \Lambda^2(T^{0,1}M) \longrightarrow T^{1,0}(M)$.

b. Prove that the operator $d^{-1,2}$ satisfies $d^{-1,2}(d\phi) = 0$ for any holomorphic function ϕ.

Exercise 3.8. Let (M, I) be an almost complex manifold. Assume that the Nijenhuis tensor $\Lambda^2(T^{1,0}M) \longrightarrow T^{0,1}(M)$ is surjective. Prove that (M, I) admits no local holomorphic functions.

Hint. Use the previous exercise.

Exercise 3.9 (*). Construct a left-invariant almost complex structure I on a Lie group G such that (G, I) admits no local holomorphic functions.

Hint. Use the previous exercise.