Complex geometry handout 4: Kähler metrics on homogeneous spaces

In this and all subsequential handouts, you are allowed to invoke the Newlander-Nirenberg theorem.

Definition 4.1. A "homogeneous almost complex manifold" is a manifold equipped with a transitive Lie group action preserving the almost complex structure. In the same way one defines objects such as "homogeneous Riemannian manifold" or "homogeneous Kähler manifold": in all these cases the Lie group acts transitively, preserving the relevant geometric structure.

Exercise 4.1. Let $(M, I) = G/H$ be a homogeneous almost complex manifold, $x \in M$, and $h \in \mathrm{St}_x(M)$ an element of the isotropy group, acting on T_xM as $2\,\mathsf{Id}$, that is, $h(x) = 2x$. Prove that I is integrable.

Exercise 4.2. Let M be a homogeneous Riemannian manifold, $x \in M$, and $h \in \mathrm{St}_x(M)$ an element of the isotropy group, acting on on T_xM as identity. Prove that h acts trivially on M.

Exercise 4.3. Construct a homogeneous, compact complex manifold M, $\dim_{\mathbb{C}} M = 2$, not admitting a Kähler metric.

Exercise 4.4. Find a complex structure on $SU(2) \times SU(2)$. Can it be Kähler?

Exercise 4.5. Find a complex structure on $SU(3)$. Can it be Kähler?

Exercise 4.6 (*). Let G be a compact, semisimple Lie group, and T its maximal torus. Assume that $\dim_{\mathbb{R}} T$ is even. Prove that G admits a left-invariant complex structure.

Exercise 4.7. Construct a $U(1, n)$-invariant metric and complex structure on $M := \frac{U(1,n)}{U(1)\times U(n)}$. Prove that it is Kähler. Prove that M is biholomorphic to an open ball in $\mathbb{C}^n$.

Remark 4.1. This metric on an open ball is called **Bergman metric**, or **complex hyperbolic metric**.

Exercise 4.8. Construct an $SO(n+2)$-invariant Kähler structure on the Grassmannian $\mathrm{Gr}_{\mathbb{R}}(2, n) := \frac{SO(n+2)}{SO(n)\times SO(2)}$.