Complex geometry handout 5: connections and torsion

Exercise 5.1. Let G be a connected Lie group, and ∇ a connection on TG such that $\nabla(X) = 0$ for all left-invariant vector fields. Prove that such ∇ exists and is unique. Prove that ∇ is torsion-free if and only if G is commutative.

Exercise 5.2. Let M be a manifold equipped with an action of quaternion algebra in TM. Prove that there exists a connection ∇ on TM such that for any quaternion $h \in \mathbb{H}$ and any $X, Y \in TM$ one has $\nabla_Y(h(X)) = h(\nabla_Y(X))$.

Exercise 5.3. Let $B \subset TM$ be a sub-bundle.

a. Prove that there exists a connection $\nabla : TM \longrightarrow TM \otimes \Lambda^1(M)$ such that $\nabla(B) \subset B \otimes \Lambda^1(M)$.

b. Suppose that ∇ is torsion-free. Prove that B is an integrable distribution, that is, $[B, B] \subset B$.

c. Assume that $[B, B] \subset B$. Prove that then ∇ can be chosen torsion-free.

Definition 5.1. Let $\mathfrak{a}(M) \subset \operatorname{End}(TM)$ be a bundle of Lie algebras. **The space of intrinsic torsion** of $\mathfrak{a}(M)$ is $\mathcal{T}_{\mathfrak{a}} := \frac{\Lambda^2(M)\otimes TM}{\mathsf{Alt}_{12}(\Lambda^1(M)\otimes\mathfrak{a}(M))}$. Let $\mathfrak{a}(M)$ be the bundle of Lie algebras leaving invariant a collection $\mathcal{A}$ of tensors or subspaces in the tensor powers of TM. Suppose that ∇ is a connection preserving $\mathcal{A}$. **The intrinsic torsion of** $\mathcal{A}$ is the class in $\mathcal{T}_{\mathfrak{a}}$ represented by the torsion of ∇.

Exercise 5.4. Let X be a nowhere vanishing vector field on M, and $\mathfrak{a}(M)$ is the bundle of Lie algebras preserving X. Prove that the intrinsic torsion space of $\mathfrak{a}(M)$ is trivial.

Exercise 5.5. Let ν be a nowhere vanishing volume form on M, and $\mathfrak{a}(M)$ is the bundle of Lie algebras preserving ν. Prove that the intrinsic torsion space of $\mathfrak{a}(M)$ is trivial. Prove that there exists a torsion-free connection preserving ν.

Exercise 5.6. Let $B \subset TM$ be a sub-bundle, and $\mathfrak{a}(M) \subset \operatorname{End}(TM)$ the Lie algebra of all $v \in \operatorname{End}(B)$ such that $v(B) \subset B$.

a. Prove that the intrinsic torsion space is isomorphic to $\Lambda^2(B^*) \otimes (TM/B)$.

b. Prove that the intrinsic torsion of B is its Frobenius form.

Definition 5.2. **Yano structure** on a manifold M is $F \in \operatorname{End}(TM)$ of constant rank satisfying $F^3 = -F$.

Exercise 5.7. Let F be a Yano structure on M.

a. Prove that there exists a connection ∇ preserving F.

b. Suppose that ∇ is torsion-free. Prove that the bundle $\mathsf{ker}\, F$ and the eigenbundle $\{x \in TM \otimes \mathbb{C} \mid F(x) = \sqrt{-1}x\}$ are integrable.