Complex geometry handout 7: Stone-Weierstrass approximation theorem

7.1 Weierstrass approximation theorem

Definition 7.1. Let M be a topological space, and $\|f\| := \sup_M |f|$ **the sup-norm on functions**. C^0-topology on the space $C^0(M)$ of continuous functions is topology defined by the sup-norm.

Exercise 7.1. Prove that C^0M with the metric defined by the sup-norm is a complete metric space.

Exercise 7.2. ("Dini's theorem")
Let $\{f_i\}$ be a sequence of bounded continuous functions on a compact space M, and suppose that $f_i(t) \geqslant f_{i-1}(t)$ for all t and i. Suppose that $\lim_i f_i(t) = f(t)$ for some continuous function f. Prove that the sequence $\{f_i(t)\}$ converges to $f(t)$ uniformly.

Exercise 7.3. Consider the sequence P_i, $i = 0, 1, 2, ...$ of polynomials on $[0, 1]$ determined recursively as follows: $P_0(t) = 0$, and $P_i(t) = P_{i-1}(t) + \frac{1}{2}(t - P_{i-1}(t)^2)$. For all $t \in [0, 1]$ and all $i = 1, 2, ...,$, prove the following.

a. Prove that $0 \leqslant P_i(t) \leqslant \sqrt{t}$.

b. Prove that $P_i(t) \geqslant P_{i-1}(t)$.

c. Prove that $\{P_i(t)\}$ converges pointwisely to $\sqrt{t}$ on $[0, 1]$.

d. Prove that $\{P_i(t)\}$ converges uniformly to $\sqrt{t}$ on $[0, 1]$

e. Prove that $Q_i(t) := P_i(t^2)$ converges uniformly to $|t|$ on $[-1, 1]$.

Exercise 7.4. Let $F(t)$ be a piecewise linear, continuous function on $[a, b] \subset \mathbb{R}$. Prove that $F(t)$ can be expressed as a sum $\sum_{i=0}^n \alpha_i |x - c_i|$ for some α_i, c_i.

Exercise 7.5. Prove that any piecewise linear, continuous function on $[a, b] \subset \mathbb{R}$ can be obtained as a uniform limit of polynomials.

Exercise 7.6. (Weierstrass approximation theorem)
Prove that any continuous function on $[a, b] \subset \mathbb{R}$ admits a uniform approximation by polynomials.

Remark 7.1. This particular proof of Weierstrass approximation is due to Lebesgue.

7.2 Stone-Weierstrass approximation theorem

From now on we assume that M is compact, Hausdorff topological space.

Definition 7.2. Let $A \subset C^0M$ be a subspace in the space of continuous functions. We say that A **separates the points** of M if for all distinct points $x, y \in M$, there exists $f \in A$ such that $f(x) \neq f(y)$.

Exercise 7.7. Let $A \subset C^0M$ be a subring, and $\bar{A}$ its closure in C^0-topology.

a. Prove that for any $a \in A$, the function $|a|$ belongs to $\bar{A}$.

b. Prove that for any $a, b \in A$, the function $\min(a, b)$ belongs to $\bar{A}$.

Hint. Use Exercise 7.3.

Exercise 7.8. Let $A \subset C^0M$ be a subring separating points, $\bar{A}$ its closure, and $U \ni x$ a neighbourhood of $x \in M$. Prove that for any $\varepsilon > 0$ there exists $a \in \bar{A}$ taking values in $[0, 1]$ such that $a(x) = 1$ and $a\Big|_{M\backslash U} < \varepsilon$.

Hint. Find a finite covering of the compact $M\backslash U$ by open sets U_i and functions $f_i \in \bar{A}$ such that $f_i(x) = 1$ and $f_i\Big|_{U\ i} < \varepsilon$, and put $a := \min_i(f_i)$.

Exercise 7.9. Let $A \subset C^0M$ be a subring separating points, $\bar{A}$ its closure, and $f \in C^0(M)$ any function. Prove that for all $x \in M$ there exists a function $f_x \in \bar{A}$ such that $f_x \leqslant f$ and $f_x(x) > f(x) - \varepsilon$.

Hint. Use the previous exercise.

Exercise 7.10. (Stone-Weierstrass theorem)
Let $A \subset C^0M$ be a subring separating points, and $\bar{A}$ its closure. Prove that $\bar{A} = C^0M$.

Hint. Use the previous exercise and find a neighbourhood U_x and a function $f_x \leqslant f$ such that $(f_x + \varepsilon)\Big|_{U_x} > f\Big|_{U_x}$. Find a finite covering $\{U_{x_i}\}$ by such U_x, such that $f \geqslant \max_i f_{x_i} > f - \varepsilon$.