Complex geometry handout 8: Plurisubharmonic functions

Exercise 8.1. Let f be a smooth real function on $\mathbb{C}^n = \mathbb{R}^{2n}$ with coordinates z_i, and $v_{2i-1} = \operatorname{Re} z_i$, $v_{2i} = \operatorname{Im} z_i$ corresponding real coordinates. Let $h := \frac{d^2 f}{dv_i dv_j}$ be the Hessian of f, considered as a bilinear symmetric form, and $g := \frac{h+I(h)}{2}$. Prove that $dd^c f(x,y) = g(Ix,y)$.

Definition 8.1. Let f be a smooth real function on a complex manifold, such that the (1,1)-form $dd^c f$ is Hermitian (hence, Kähler). Then f is called **strictly plurisubharmonic.** In that case f is called **the Kähler potential** of the Kähler form $dd^c f$. If all eigenvalues of $dd^c f$ are non-negative, f is called **plurisubharmonic**.

Exercise 8.2. Prove that any smooth convex function on $\mathbb{C}^n$ is plurisubharmonic.

Exercise 8.3. Let f be a smooth function on $\mathbb{R}^n$, and $\phi \in C^\infty(\mathbb{C}^n)$ map $(z_1, ..., z_n)$ to $f(\operatorname{Re} z_1, ..., \operatorname{Re} z_n)$. Prove that ϕ is plurisubharmonic if and only if f is convex.

Exercise 8.4. Let ω be a Kähler form. Using the Poincaré-Dolbeault-Grothendieck lemma, prove that ω locally admits a Kähler potential.

Exercise 8.5. Let f be plurisubharmonic. Prove that e^f is also plurisubharmonic.

Exercise 8.6. Let f be a plurisubharmonic function satisfying $f < 0$. Prove that $\log(-f)$ is also plurisubharmonic.

Exercise 8.7. Let f be a strictly plurisubharmonic function. Prove that f cannot have a maximum.

Exercise 8.8. Let f be a plurisubharmonic function. Prove that f^2 is also plurisubharmonic or find a counterexample.