Complex geometry

lecture 1: complex manifolds

Misha Verbitsky

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Complex structure on a vector space

DEFINITION: Let V be a vector space over $\mathbb{R}$, and $I: V \longrightarrow V$ an automorphism which satisfies $I^2 = -\operatorname{Id}_V$. Such an automorphism is called **a complex structure operator** on V.

We extend the action of I on the tensor spaces $V \otimes V \otimes ... \otimes V \otimes V^* \otimes V^* \otimes ... \otimes V^*$ by multiplicativity: $I(v_1 \otimes ... \otimes w_1 \otimes ... \otimes w_n) = I(v_1) \otimes ... \otimes I(w_1) \otimes ... \otimes I(w_n)$.

Trivial observations:

1. **The eigenvalues α_i of I are $\pm\sqrt{-1}$.** Indeed, $\alpha_i^2 = -1$.

2. **V admits an I-invariant, positive definite scalar product ("metric") g.** Take any metric g_0, and let $g := g_0 + I(g_0)$.

3. **I is orthogonal for such g.**
Indeed, $g(Ix, Iy) = g_0(x, y) + g_0(Ix, Iy) = g(x, y)$.

4. **I diagonalizable over $\mathbb{C}$.** Indeed, any orthogonal matrix is diagonalizable.

5. **There are as many $\sqrt{-1}$-eigenvalues as there are $-\sqrt{-1}$-eigenvalues.**

Complex structure operator in coordinates

This implies that in an appropriate basis in $V \otimes_{\mathbb{R}} \mathbb{C}$, the complex structure operator is diagonal, as follows:

$$\left[\begin{array}{cccc|cccc}
\sqrt{-1} & & & & & & & \\
& \sqrt{-1} & & & & & 0 & \\
& & \ddots & & & & & \\
& & & \sqrt{-1} & & & & \\
\hline
& & & & -\sqrt{-1} & & & \\
& 0 & & & & -\sqrt{-1} & & \\
& & & & & & \ddots & \\
& & & & & & & -\sqrt{-1}
\end{array}\right]$$

We also obtain its normal form in a real basis:

$$\begin{bmatrix}
0 & -1 & & & & & & \\
1 & 0 & & & & & & \\
& & 0 & -1 & & & & \\
& & 1 & 0 & & & & \\
& & & & \ddots & & & \\
& & & & & \ddots & & \\
& & & & & & 0 & -1 \\
& & & & & & 1 & 0
\end{bmatrix}$$

Hodge decomposition

DEFINITION: Let (V, I) be a space equipped with a complex structure. **The Hodge decomposition** $V \otimes_{\mathbb{R}} \mathbb{C} := V^{1,0} \oplus V^{0,1}$ is defined in such a way that $V^{1,0}$ is a $\sqrt{-1}$-eigenspace of I, and $V^{0,1}$ a $-\sqrt{-1}$-eigenspace.

REMARK: In the same way one defines the Hodge decomposition on the dual space V^*.

Remark 1: The space $V^{1,0} \subset V \otimes_{\mathbb{R}} \mathbb{C}$ **uniquely determines the operator I.** Indeed, $I = \sqrt{-1}$ on $V^{1,0}$ and $I = -\sqrt{-1}$ on $V^{0,1}$. This gives **a bijection between the set of complex structures on V and the set of subspaces $W \subset V \otimes_{\mathbb{R}} \mathbb{C}$ such that $\dim_{\mathbb{C}} W = \frac{1}{2} \dim_{\mathbb{R}} V$ and $W \cap \overline{W} = 0$.**

Hermitian structures

DEFINITION: Let (V, I) be a space equipped with a complex structure. **The Hodge decomposition** $V \otimes_{\mathbb{R}} \mathbb{C} := V^{1,0} \oplus V^{0,1}$ is defined in such a way that $V^{1,0}$ is a $\sqrt{-1}$-eigenspace of I, and $V^{0,1}$ a $-\sqrt{-1}$-eigenspace.

DEFINITION: An I-invariant positive definite scalar product on (V, I) is called **an Hermitian metric**, and (V, I, g) – an Hermitian space.

REMARK: Let I be a complex structure operator on a real vector space V, and g – a Hermitian metric. Then **the bilinear form $\omega(x, y) := g(x, Iy)$ is skew-symmetric.** Indeed, $\omega(x, y) = g(x, Iy) = g(Ix, I^2 y) = -g(Ix, y) = -\omega(y, x)$.

DEFINITION: A skew-symmetric form $\omega(x, y)$ is called **an Hermitian form on (V, I)**.

REMARK: In the triple I, g, ω, **each element can recovered from the other two.**

Holomorphic functions

DEFINITION: Let $I : TM \longrightarrow TM$ be an endomorphism of a tangent bundle satisfying $I^2 = -\operatorname{Id}$. Then I is called **almost complex structure operator**, and the pair (M, I) **an almost complex manifold**.

EXAMPLE: $M = \mathbb{C}^n$, with complex coordinates $z_i = x_i + \sqrt{-1}\, y_i$, and $I(d/dx_i) = d/dy_i$, $I(d/dy_i) = -d/dx_i$.

EXAMPLE: In complex dimension 1, **almost complex structure is the same as conformal structure with orientation (prove it).**

DEFINITION: A function $f : M \longrightarrow \mathbb{C}$ on an almost complex manifold is called **holomorphic** if $df \in \Lambda^{1,0}(M)$.

REMARK: For some almost complex manifolds, **there are no holomorphic functions at all**, even locally.

Example: S^6 with the unique G_2-invariant almost complex structure.

Holomorphic functions on $\mathbb{C}^n$

THEOREM: Let $f: M \longrightarrow \mathbb{C}$ be a differentiable function on an open subset $M \subset \mathbb{C}^n$, with almost complex structure as above. **Then TFAE:**
(1) f **is holomorphic**.
(2) The differential $df: TM \longrightarrow \mathbb{C}$, considered as a form on the vector space $T_xM = T_x\mathbb{C}^n = \mathbb{C}^n$ **is $\mathbb{C}$-linear.**
(3) For any complex affine line $L \in \mathbb{C}^n$, the restriction $f|_L = \mathbb{C}$ is **holomorphic (complex analytic) as a function of one complex variable.**
(4) f is expressed as a sum of Taylor series around any point $(z_1, ..., z_n) \in M$.

Proof: (1) and (2) are tautologically equivalent. Equivalence of (1) and (3) is also clear, because a restriction of $\theta \in \Lambda^{1,0}(M)$ to a line is a $(1,0)$-form on a line, and, conversely, if df is of type (1,0) on each complex line, it is of type (1,0) on TM, which is implied by the following linear-algebraic observation.

LEMMA: Let $\eta \in V^* \otimes \mathbb{C}$ be a complex-valued linear form on a vector space (V, I) equipped with a complex structure. **Then $\eta \in \Lambda^{1,0}(V)$ if and only if its restriction to any I-invariant 2-dimensional subspace L belongs to $\Lambda^{1,0}(L)$.**

EXERCISE: Prove it.

(4) clearly implies (2). (1) implies (4) by Cauchy formula.

Sheaves

DEFINITION: A **presheaf of functions** on a topological space M is a collection of subrings $\mathcal{F}(U) \subset C(U)$ in the ring $C(U)$ of all functions on U, for each open subset $U \subset M$, such that the restriction of every $\gamma \in \mathcal{F}(U)$ to an open subset $U_1 \subset U$ belongs to $\mathcal{F}(U_1)$.

DEFINITION: A presheaf of functions $\mathcal{F}$ is called **a sheaf of functions** if these subrings satisfy the following condition. Let $\{U_i\}$ be a cover of an open subset $U \subset M$ (possibly infinite) and $f_i \in \mathcal{F}(U_i)$ a family of functions defined on the open sets of the cover and compatible on the pairwise intersections:

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$

for every pair of members of the cover. **Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.**

Sheaves and exact sequences

REMARK: A presheaf of functions is a collection of subrings of functions on open subsets, compatible with restrictions. **A sheaf of fuctions is a presheaf allowing "gluing"** a function on a bigger open set if its restrictions to smaller open sets are compatible.

DEFINITION: A sequence $A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow ...$ of homomorphisms of abelian groups or vector spaces is called **exact** if the image of each map is the kernel of the next one.

CLAIM: A presheaf $\mathcal{F}$ is a sheaf if and only if for every cover $\{U_i\}$ of an open subset $U \subset M$, **the sequence of restriction maps**

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{i \neq j} \mathcal{F}(U_i \cap U_j)$$

is exact, with $\eta \in \mathcal{F}(U_i)$ mapped to $\eta\big|_{U_i \cap U_j}$ and $-\eta\big|_{U_j \cap U_i}$.

Sheaves and presheaves: examples

Examples of sheaves:

* Space of continuous functions

* Space of smooth functions, any differentiability class

* Space of real analytic functions

Examples of presheaves which are not sheaves:

* Space of constant functions **(why?)**

* Space of bounded functions **(why?)**

Ringed spaces

A **ringed space** $(M, \mathcal{F})$ is a topological space equipped with a sheaf of functions. A **morphism** $(M, \mathcal{F}) \xrightarrow{\Psi} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \xrightarrow{\Psi} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $\psi^* f := f \circ \Psi$ belongs to the ring $\mathcal{F}\Big(\Psi^{-1}(U)\Big)$. An **isomorphism** of ringed spaces is a homeomorphism Ψ such that Ψ and Ψ^{-1} are morphisms of ringed spaces.

EXAMPLE: Let M be a manifold of class C^i and let $C^i(U)$ be the space of functions of this class. **Then C^i is a sheaf of functions, and (M, C^i) is a ringed space.**

REMARK: Let $f: X \longrightarrow Y$ be a smooth map of smooth manifolds. Since a pullback $f^*\mu$ of a smooth function $\mu \in C^\infty(M)$ is smooth, **a smooth map of smooth manifolds defines a morphism of ringed spaces.**

Converse is also true:

Ringed spaces and smooth maps

CLAIM: Let (M, C^i) and (N, C^i) be manifolds of class C^i. Then **there is a bijection between smooth maps $f: M \longrightarrow N$ and the morphisms of corresponding ringed spaces.**

Proof: Any smooth map induces a morphism of ringed spaces. Indeed, **a composition of smooth functions is smooth, hence a pullback is also smooth.**

Conversely, let $U_i \longrightarrow V_i$ be a restriction of f to some charts; to show that f is smooth, it would suffice to show that $U_i \longrightarrow V_i$ is smooth. However, we know that a pullback of any smooth function is smooth. **Therefore, Claim is implied by the following lemma.**

LEMMA: Let M, N be open subsets in $\mathbb{R}^n$ and let $f: M \to N$ map such that a pullback of any function of class C^i belongs to C^i. **Then f is of class C^i.**

Proof: Apply f to coordinate functions. ■

Smooth manifolds defined through sheaves

As we have just shown, this definition is equivalent to the previous one.

DEFINITION: Let $(M, \mathcal{F})$ be a topological manifold equipped with a sheaf of functions. It is said to be a **smooth manifold of class** C^∞ or C^i if every point in $(M, \mathcal{F})$ has an open neighborhood isomorphic to the ringed space $(\mathbb{R}^n, \mathcal{F}')$, where $\mathcal{F}'$ is a ring of functions on $\mathbb{R}^n$ of this class.

DEFINITION: A chart, or **a coordinate system** on an open subset U of a manifold $(M, \mathcal{F})$ is an isomorphism between $(U, \mathcal{F})$ and an open subset in $(\mathbb{R}^n, \mathcal{F}')$, where $\mathcal{F}'$ are functions of the same class on $\mathbb{R}^n$.

DEFINITION: Diffeomorphism of smooth manifolds is a homeomorphism φ which induces an isomorphim of ringed spaces, that is, φ and φ^{-1} map (locally defined) smooth functions to smooth functions.

Complex manifolds

DEFINITION: A holomorphic function on $\mathbb{C}^n$ is a function $f: \mathbb{C}^n \longrightarrow \mathbb{C}$ such that df is complex linear, that is $df \in \Lambda^{1,0}(M)$.

REMARK: Holomorphic functions form a sheaf.

DEFINITION: A complex manifold M is a ringed space which is locally isomorphic to an open ball in $\mathbb{C}^n$ with a sheaf of holomorphic functions.

REMARK: In other words, M **is covered with open balls embedded to** $\mathbb{C}^n$ and transition functions (being coordinate functions for one ball considered in other coordinate system) **are holomorphic.**

Complex manifolds and almost complex manifolds

DEFINITION: Standard almost complex structure is $I(d/dx_i) = d/dy_i$, $I(d/dy_i) = -d/dx_i$ on $\mathbb{C}^n$ with complex coordinates $z_i = x_i + \sqrt{-1}\, y_i$.

DEFINITION: A map $\Psi : (M, I) \longrightarrow (N, J)$ from an almost complex manifold to an almost complex manifold is called **holomorphic** if $\Psi^*(\Lambda^{1,0}(N)) \subset \Lambda^{1,0}(M)$.

REMARK: This is the same as $d\Psi$ being complex linear; for standard almost complex structures, **this is the same as the coordinate components of Ψ being holomorphic functions.**

Another definition: A complex manifold is a manifold equipped with an atlas with charts identified with open subsets of $\mathbb{C}^n$ and transition functions holomorphic.

EXERCISE: Prove that **this definition is equivalent to the one with sheaves**.

Integrability of almost complex structures

DEFINITION: An almost complex structure I on a manifold is called **integrable** if any point of M has a neighbourhood U diffeomorphic to an open subset of $\mathbb{C}^n$, in such a way that the almost complex structure I is induced by the standard one on $U \subset \mathbb{C}^n$.

CLAIM: Complex structure on a manifold M uniquely determines an integrable almost complex structure, and is determined by it.

Proof: Complex structure on a manifold M is determined by the sheaf of holomorphic functions $\mathcal{O}_M$, and $\mathcal{O}_M$ is determined by I as explained above. Therefore, an integrable almost complex structure defines a complex structure. Conversely, every complex structure gives a sub-bundle in $\Lambda^{1,0}(M) = d\mathcal{O}_M \subset \Lambda^1(M, \mathbb{C})$, and **such a sub-bundle defines an almost complex structure by Remark 1.** ∎