Complex geometry

lecture 2: Cauchy formula

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The Grassmann algebra

DEFINITION: Let V be a vector space. Denote by $\Lambda^i V$ the space of antisymmetric polylinear i-forms on V^*, and let $\Lambda^* V := \bigoplus \Lambda^i V$. Denote by $T^{\otimes i} V$ the algebra of **all** polylinear i-forms on V^* ("tensor algebra"), and let $\mathsf{Alt}: T^{\otimes i} V \longrightarrow \Lambda^i V$ be **the antisymmetrization**,

$$\mathsf{Alt}(\eta)(x_1, ..., x_i) := \frac{1}{i!} \sum_{\sigma \in \Sigma_i} (-1)^{\tilde{\sigma}} \eta(x_{\sigma_1}, ..., x_{\sigma_i})$$

where Σ_i is the group of permutations, and $\tilde{\sigma} = 1$ for odd permutations, and 0 for even. Consider the multiplicative operation ("wedge-product") on $\Lambda^* V$, denoted by $\eta \wedge \nu := \mathsf{Alt}(\eta \otimes \nu)$. The space $\Lambda^* V$ with this operation is called **the Grassmann algebra**.

REMARK: It is an algebra of anti-commutative polynomials.

Properties of Grassmann algebra:

1. $\dim \Lambda^i V := \binom{\dim V}{i}$, $\dim \Lambda^* V = 2^{\dim V}$.

2. $\Lambda^*(V \oplus W) = \Lambda^*(V) \otimes \Lambda^*(W)$.

The Hodge decomposition in linear algebra

DEFINITION: Let (V, I) be a space equipped with a complex structure. **The Hodge decomposition** $V \otimes_{\mathbb{R}} \mathbb{C} := V^{1,0} \oplus V^{0,1}$ is defined in such a way that $V^{1,0}$ is a $\sqrt{-1}$-eigenspace of I, and $V^{0,1}$ a $-\sqrt{-1}$-eigenspace.

REMARK: Let $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. The Grassmann algebra of skew-symmetric forms $\Lambda^n V_{\mathbb{C}} := \Lambda^n_{\mathbb{R}} V \otimes_{\mathbb{R}} C$ admits a decomposition

$$\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}$$

We denote $\Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}$ by $\Lambda^{p,q} V$. The resulting decomposition $\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^{p,q} V$ is called **the Hodge decomposition of the Grassmann algebra**.

REMARK: The operator I induces $U(1)$-action on V by the formula $\rho(t)(v) = \cos t \cdot v + \sin t \cdot I(v)$. We extend this action on the tensor spaces by muptiplicativity.

$U(1)$-representations and the weight decomposition

REMARK: Any complex representation W of $U(1)$ is written as a sum of 1-dimensional representations $W_i(p)$, with $U(1)$ acting on each $W_i(p)$ as $\rho(t)(v) = e^{\sqrt{-1}\,pt}(v)$. The 1-dimensional representations are called **weight p representations of $U(1)$.**

DEFINITION: A **weight decomposition** of a $U(1)$-representation W is a decomposition $W = \oplus W^p$, where each $W^p = \oplus_i W_i(p)$ is a sum of 1-dimensional representations of weight p.

REMARK: The Hodge decomposition $\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^{p,q} V$ is a weight decomposition, with $\Lambda^{p,q}V$ being a weight $p-q$-component of $\Lambda^n V_{\mathbb{C}}$.

REMARK: $V^{p,p}$ is the space of $U(1)$-invariant vectors in $\Lambda^{2p}V$.

Further on, **TM is the tangent bundle on a manifold, and $\Lambda^i M$ the space of differential i-forms.** It is a Grassmann algebra on TM.

Vector fields

DEFINITION: Let X be the vector field on a manifold M, and f a function. Denote by $\operatorname{Lie}_X f$ **the derivatiive** of f along X.

DEFINITION: **A derivation** on a commutative ring is a map $R \xrightarrow{d} R$ satisfying **the Leibniz identity** $d(xy) = d(x)y + xd(y)$.

THEOREM: Each derivation of the ring $C^\infty M$ of smooth functions on M is given by a vector field X; **this correspondence is bijective.**

REMARK: This can be used as a definition of a vector field.

EXERCISE: Prove that **a commutator of two derivations is again a derivation.**

REMARK: Vector fields are the same as derivations of $C^\infty M$. This allows us to define **the commutator of two vector fields** as the commutator of the corresponding derivations.

DEFINITION: Denote by TM the bundle of vector fields, and by $\Lambda^1 M$ or T^* the dual bundle, called **the bundle of 1-forms**. For any $f \in C^\infty M$, the operation $X \longrightarrow \operatorname{Lie}_X f$ is linear as a function of X, hence it defines a section of T^*M. We denote this section df, and call it **the differential** of f.

De Rham algebra

DEFINITION: Let $\Lambda^* M$ denote the vector bundle with the fiber $\Lambda^* T_x^* M$ at $x \in M$ ($\Lambda^* T^* M$ is the Grassman algebra of the cotangent space $T_x^* M$). The sections of $\Lambda^i M$ are called **differential i-forms**. The algebraic operation "wedge product" defined on differential forms is $C^\infty M$-linear; the space $\Lambda^* M$ of all differential forms is called **the de Rham algebra**.

REMARK: $\Lambda^0 M = C^\infty M$.

THEOREM: **There exists a unique operator $C^\infty M \xrightarrow{d} \Lambda^1 M \xrightarrow{d} \Lambda^2 M \xrightarrow{d} \Lambda^3 M \xrightarrow{d} \dots$ satisfying the following properties**

1. On functions, d is equal to the differential.
2. $d^2 = 0$
3. $d(\eta \wedge \xi) = d(\eta) \wedge \xi + (-1)^{\tilde\eta} \eta \wedge d(\xi)$, where $\tilde\eta = 0$ where $\eta \in \lambda^{2i} M$ is **an even form,** and $\eta \in \lambda^{2i+1} M$ is **odd.**

DEFINITION: The operator d is called **de Rham differential**.

EXERCISE: **Prove it**.

DEFINITION: A form η is called **closed** if $d\eta = 0$, **exact** if $\eta \in \operatorname{im} d$. The group $\frac{\ker d}{\operatorname{im} d}$ is called **de Rham cohomology** of M.

Cauchy formula in dimension 1 (statement)

DEFINITION: Let $U \subset \mathbb{C}^n$ be an open subset, and $f : U \longrightarrow \mathbb{C}$ a function of class C^1 (differentiable at least once). We say that f is **holomorphic** if the differential $df : T_xU \longrightarrow \mathbb{C}$ is complex linear at each $x \in U$.

REMARK: Clearly, f is holomorphic if and only if $df \in \Lambda^{1,0}(U)$, where $\Lambda^{1,0}(U)$ is the Hodge (1,0)-component of the de Rham algebra.

Taylor series decomposition for holomorphic functions in 1 variable is implied by the Cauchy formula: for any folomorphic function f in disk $\Delta \subset \mathbb{C}$,

$$\int_{\partial\Delta} \frac{f(z)dz}{z-a} = 2\pi\sqrt{-1}\, f(a),$$

where $a \in \Delta$ any point, and z coordinate on $\mathbb{C}$. Indeed, in this case,

$$2\pi\sqrt{-1}\, f(a) = \sum_{i \geqslant 0} a^i \int_{\partial\Delta} f(z)(z^{-1})^{i+1},$$

because $\frac{1}{z-a} = z^{-1} \sum_{i\geqslant 0}(az^{-1})^i$.

Cauchy formula in dimension 1 (proof)

Let's prove Cauchy formula, using Stokes' theorem. Since the space $\Lambda^{1,0}\mathbb{C}$ is 1-dimensional, $df \wedge dz = 0$ for any holomorphic function on $\mathbb{C}$. This gives

CLAIM: A function on a disk $\Delta \subset \mathbb{C}$ **is holomorphic if and only if the form** $\eta := f dz$ **is closed** (that is, satisfies $d\eta = 0$). ■

Now, let S_ε be a radius ε circle around a point $a \in \Delta$, Δ_ε its interior, and $\Delta_0 := \Delta \backslash \Delta_\varepsilon$. Stokes' theorem gives

$$0 = \int_{\Delta_0} d\left(\frac{f(z)dz}{z-a}\right) = -\int_{S_\varepsilon} \frac{f(z)dz}{z-a} + \int_{\partial\Delta} \frac{f(z)dz}{z-a},$$

hence Cauchy formula would follow if we show that $\lim_{\varepsilon \to 0} \int_{S_\varepsilon} \frac{f(z)dz}{z-a} = 2\pi\sqrt{-1} f(a)$.

Assuming for simplicity $a = 0$ and parametrizing the circle S_ε by $\varepsilon e^{\sqrt{-1}\,t}$, we obtain

$$\int_{S_\varepsilon} \frac{f(z)dz}{z} = \int_0^{2\pi} \frac{f(\varepsilon e^{\sqrt{-1}\,t})}{\varepsilon e^{\sqrt{-1}\,t}} d(\varepsilon e^{\sqrt{-1}\,t}) =$$

$$= \int_0^{2\pi} \frac{f(\varepsilon e^{\sqrt{-1}\,t})}{\varepsilon e^{\sqrt{-1}\,t}} \sqrt{-1}\,\varepsilon e^{\sqrt{-1}\,t} dt = \int_0^{2\pi} f(\varepsilon e^{\sqrt{-1}\,t}) \sqrt{-1}\,dt$$

as ε tends to 0, $f(\varepsilon e^{\sqrt{-1}\,t})$ tends to $f(0)$, and this integral goes to $2\pi\sqrt{-1} f(0)$.

Holomorphic functions on $\mathbb{C}^n$ (reminder)

THEOREM: Let $f: U \longrightarrow \mathbb{C}$ be a differentiable function on an open subset $U \subset \mathbb{C}^n$. **Then the following are equivalent.**
(1) f **is holomorphic**.
(2) For any complex affine line $L \in \mathbb{C}^n$, the restriction $f|_L = \mathbb{C}$ is **holomorphic as a function of one complex variable.**
(3) f is expressed as a sum of Taylor series around any point $(z_1, ..., z_n) \in U$: for all sufficiently small $t_1, ..., t_n$, one has $f(z_1 + t_1, z_2 + t_2, ..., z_n + t_n) = \sum_{i_1,...,i_n} a_{i_1,...,i_n} t_1^{i_1} t_2^{i_2} ... t_n^{i_n}$.

Proof: Equivalence of (1) and (2) is clear, because a restriction of $\theta \in \Lambda^{1,0}(M)$ to a line is a $(1,0)$-form on a line, and, conversely, if df is of type (1,0) on each complex line, it is of type (1,0) on TM, which is implied by the following linear-algebraic observation.

LEMMA: Let $\eta \in V^* \otimes \mathbb{C}$ be a complex-valued linear form on a real vector space (V, I) equipped with a complex structure I. **Then $\eta \in \Lambda^{1,0}(V)$ if and only if its restriction to any I-invariant 2-dimensional subspace L belongs to $\Lambda^{1,0}(L)$.**
EXERCISE: Prove it.

(3) clearly implies (1). (1) implies (3) by Cauchy formula (many variables), proven below.

Cauchy formula (many variables)

REMARK: Let $U \subset \mathbb{C}^n$ be an open subset, and $z_1, ..., z_n$ complex coordinates. Holomorphicity of $f: U \longrightarrow \mathbb{C}$ is equivalent to $df \in \Lambda^{1,0}(M)$, which is equivalent to $df \wedge dz_1 \wedge dz_1 \wedge ... \wedge dz_n = 0$. Denote the form $dz_1 \wedge dz_1 \wedge ... \wedge dz_n$ by Φ. We obtain that **f is holomorphic if and only if the form $f\Phi$ is closed**

THEOREM: (Cauchy formula in dimension n)
Let $\Delta \subset \mathbb{C}^n$ be a polydisk (product of disks) of radius 1, and $\alpha_1, ..., \alpha_n \in \Delta$ complex numbers. Denote by $S \subset \mathbb{C}^n$ the product of circles of radius 1 in variables $z_1, ..., z_n$:, $S = S_1(z_1) \times S_1(z_2) \times ... \times S_1(z_n)$. Let f be a holomorphic function in a polydisk. **Then $\int_S V = (2\pi\sqrt{-1}\,)^n f(\alpha_1, ...\alpha_n)$, where**

$$V = \frac{f\Phi}{(z_1 - \alpha_1)(z_2 - \alpha_2) \times ... \times (z_n - \alpha_n)}.$$

Proof. Step 1: Denote by Z the set $\bigcup_{i=1}^n \{(z_1, ..., z_n) \mid z_i = \alpha_i\}$. The complement of Z is the set of definition of the closed differential form V. Let S_ε be the product of circles of radius ε with center in $\alpha_1, ..., \alpha_n$. Then $S, S_\varepsilon \subset \mathbb{C}^n \backslash Z$, and **the tori S, S_ε are homotopy equivalent in the domain $\mathbb{C}^n \backslash Z$, where V is closed. It remains to show that $\lim_{\varepsilon \to 0} \int_{S_\varepsilon} V = (2\pi\sqrt{-1}\,)^n f(\alpha_1, ...\alpha_n)$.**

Cauchy formula (many variables), part 2

THEOREM: (Cauchy formula in dimension n)
Let $\Delta \subset \mathbb{C}^n$ be a polydisk (product of disks) of radius 1, and $\alpha_1, ..., \alpha_n \in \Delta$ complex numbers. Denote by $S \subset \mathbb{C}^n$ the product of circles of radius 1 in variables $z_1, ..., z_n$:, $S = S_1(z_1) \times S_1(z_2) \times ... \times S_1(z_n)$. Let f be a holomorphic function in a polydisk. **Then $\int_S V = (2\pi\sqrt{-1}\,)^n f(\alpha_1, ...\alpha_n)$, where**

$$V = \frac{f\Phi}{(z_1-\alpha_1)(z_2-\alpha_2)...(z_n-\alpha_n)}.$$

Proof. Step 1: Let S_ε be a product of circles of radius ε with center in $\alpha_1, ..., \alpha_n$. **It remains to show that $\lim_{\varepsilon\to 0}\int_{S_\varepsilon} V = (2\pi\sqrt{-1}\,)^n f(\alpha_1, ...\alpha_n)$.**

Step 2: To simplify notation we set $\alpha_i = 0$. Parametrize S_ε by the cube $[0, 2\pi]^n$ using the map $t_1, ..., t_n \longrightarrow \varepsilon e^{\sqrt{-1}\,t_1}, ..., \varepsilon e^{\sqrt{-1}\,t_n}$. This gives

$$\int_{S_\varepsilon} V = \int_{S_\varepsilon} f(z)\frac{dz_1}{z_1}\wedge ...\wedge \frac{dz_n}{z_n} =$$

$$= \int_0^{2\pi} ... \int_0^{2\pi} \frac{f(\varepsilon e^{\sqrt{-1}\,t_1}, \varepsilon e^{\sqrt{-1}\,t_2}, ..., \varepsilon e^{\sqrt{-1}\,t_n})}{\varepsilon e^{\sqrt{-1}\,t_1}\varepsilon e^{\sqrt{-1}\,t_2}...\varepsilon e^{\sqrt{-1}\,t_n}}\varepsilon^n d\left(e^{\sqrt{-1}\,t_1}\right) d\left(e^{\sqrt{-1}\,t_2}\right) ... d\left(e^{\sqrt{-1}\,t_n}\right) =$$

$$= (\sqrt{-1}\,)^n \int_0^{2\pi} ... \int_0^{2\pi} f(\varepsilon e^{\sqrt{-1}\,t_1}, ..., \varepsilon e^{\sqrt{-1}\,t_n}) dt_1 dt_2 ... dt_n,$$

which converges to $(2\pi\sqrt{-1}\,)^n f(0, ..., 0)$. ■

Cauchy formula and Taylor expansion

REMARK: Cauchy formula implies that **holomorphic functions defined in a polydisk have Taylor expansion in this polydisk**. Indeed,

$$f(\alpha_1, ...\alpha_n) = \frac{1}{(2\pi\sqrt{-1}\,)^n} \int_S \frac{f dz_1 \wedge ... \wedge dz_n}{(z_1 - \alpha_1)(z_2 - \alpha_2) \times ... \times (z_n - \alpha_n)}$$

Take the Taylor expansion of $(z_i - \alpha_i)^{-1}$ using

$$\frac{1}{(z_i - \alpha_i)} = \frac{z_i^{-1}}{(1 - \alpha_i z_i^{-1})} = \sum_{l=0}^{\infty} \alpha_i^l z_i^{-l-1}.$$

Then

$$f(\alpha_1, ...\alpha_n) = \sum_{i_1=0}^{\infty} ... \sum_{i_n=0}^{\infty} \alpha_1^{i_1} .... \alpha_{i_n}^{i_n} \int_S f(z_1, ..., z_n) z_1^{-i_1-1} ... z_n^{-i_n-1} dz_1 \wedge ... \wedge dz_n.$$

Complex manifolds (reminder)

DEFINITION: A holomorphic function on $\mathbb{C}^n$ is a function $f: \mathbb{C}^n \longrightarrow \mathbb{C}$ such that df is complex linear, that is $df \in \Lambda^{1,0}(M)$.

REMARK: Holomorphic functions form a sheaf.

DEFINITION: A complex manifold M is a ringed space which is locally isomorphic to an open ball in $\mathbb{C}^n$ with a sheaf of holomorphic functions.

REMARK: In other words, M **is covered with open balls embedded to** $\mathbb{C}^n$ and transition functions (being coordinate functions for one ball considered in other coordinate system) **are holomorphic.**

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Integrability of almost complex structures

DEFINITION: An almost complex structure I on a manifold is called **integrable** if any point of M has a neighbourhood U diffeomorphic to an open subset of $\mathbb{C}^n$, in such a way that the almost complex structure I is induced by the standard one on $U \subset \mathbb{C}^n$.

CLAIM: Complex structure on a manifold M uniquely determines an integrable almost complex structure, and is determined by it.

Proof: Complex structure on a manifold M is determined by the sheaf of holomorphic functions $\mathcal{O}_M$, and $\mathcal{O}_M$ is determined by I as explained above. Therefore, an integrable almost complex structure defines a complex structure. Conversely, every complex structure gives a sub-bundle in $\Lambda^{1,0}(M) = d\mathcal{O}_M \subset \Lambda^1(M, \mathbb{C})$, and **such a sub-bundle defines an almost complex structure by Remark 1.** ■

Frobenius form

CLAIM: Let $B \subset TM$ be a sub-bundle of a tangent bundle of a smooth manifold. Given vector fiels $X, Y \in B$, consider their commutator $[X, Y]$, and lets $\Psi(X, Y) \in TM/B$ be the projection of $[X, Y]$ to TM/B. **Then $\Psi(X, Y)$ is $C^\infty(M)$-linear in X, Y:**

$$\Psi(fX, Y) = \Psi(X, fY) = f\Psi(X, Y).$$

Proof: Leibnitz identity gives $[X, fY] = f[X, Y] + X(f)Y$, and the second term belongs to B, hence does not influence the projection to TM/B. ■

DEFINITION: This form is called **the Frobenius form** of the sub-bundle $B \subset TM$. This bundle is called **involutive**, or **integrable**, or **holonomic** if $\Psi = 0$.

EXERCISE: Give an example of a non-integrable sub-bundle.

Formal integrability

DEFINITION: An almost complex structure I on (M, I) is called **formally integrable** if $[T^{1,0}M, T^{1,0}] \subset T^{1,0}$, that is, if $T^{1,0}M$ is involutive.

DEFINITION: The Frobenius form $\Psi \in \Lambda^2(\Lambda^{1,0}M) \otimes T^{0,1}M$ is called **the Nijenhuis tensor**.

CLAIM: If a complex structure I on M is integrable, it is formally integrable.

Proof: Locally, the bundle $T^{1,0}(M)$ is generated by d/dz_i, where z_i are complex coordinates. These vector fields commute, hence satisfy $[d/dz_i, d/dz_j] \in T^{1,0}(M)$. This means that the Frobenius form vanishes. ■

THEOREM: (Newlander-Nirenberg)
A complex structure I on M is integrable if and only if it is formally integrable.

Proof: (real analytic case) next lecture, probably.

REMARK: In dimension 1, formal integrability is automatic. Indeed, $T^{1,0}M$ is 1-dimensional, hence all skew-symmetric 2-forms on $T^{1,0}M$ vanish.

Possible topics for the next lectures

1. Proof of Frobenius theorem.

2. Newlander-Nirenberg theorem for real analytic almost complex manifolds.