Complex geometry

lecture 3: Frobenius theorem

Misha Verbitsky

HSE, room 306, 16:20,

September 30, 2020

Distributions (reminder)

DEFINITION: Distribution on a manifold is a sub-bundle $B \subset TM$

REMARK: Let $\Pi: TM \longrightarrow TM/B$ be the projection, and $x, y \in B$ some vector fields. Then $[fx, y] = f[x, y] - D_y(f)x$. This implies that **$\Pi([x, y])$ is $C^\infty(M)$-linear as a function of x and y.**

DEFINITION: The map $[B, B] \longrightarrow TM/B$ we have constructed is called **Frobenius bracket** (or **Frobenius form**); it is a skew-symmetric $C^\infty(M)$-linear form on B with values in TM/B.

DEFINITION: A distribution is called **integrable**, or **holonomic**, or **involutive**, if its Frobenius form vanishes.

Smooth submersions

DEFINITION: Let $\pi : M \longrightarrow M'$ be a smooth map of manifolds. This map is called **submersion** if at each point of M the differential $D\pi$ is surjective, and **immersion** if it is injective.

CLAIM: Let $\pi : M \longrightarrow M'$ be a submersion. Then each $m \in M$ has a neighbourhood $U \cong V \times W$, where V, W are smooth and $\pi|_U$ **is a projection of $V \times W = U \subset M$ to $W \subset M'$ along V.**

EXERCISE: Deduce this result from the inverse function theorem.

EXERCISE: ("Ehresmann's fibration theorem")
Let $\pi : M \longrightarrow M'$ be a smooth submersion of compact manifolds. Prove that π is a locally trivial fibration.

DEFINITION: Vertical tangent space $T_\pi M \subset TM$ of a submersion $\pi : M \longrightarrow M'$ is the kernel of $D\pi$.

CLAIM: Let $\pi : M \longrightarrow M'$ be a submersion and $T_\pi M \subset TM$ the vertical tangent space. **Then $T_\pi M$ is an involutive subbundle.**

Proof: $D_\pi([X,Y]) = [D_\pi(X), D_\pi(Y)] = 0$ for any $X, Y \in \ker D_\pi$. ∎

Frobenius theorem (statement)

Frobenius Theorem: Let $B \subset TM$ be a sub-bundle. Then B is involutive if and only if each point $x \in M$ has a neighbourhood $U \ni x$ and **a smooth submersion $U \xrightarrow{\pi} V$ such that B is its vertical tangent space: $B = T_\pi M$.**

REMARK: The implication **"$B = T_\pi M$" $\Rightarrow$ "Frobenius form vanishes"** was proven above.

DEFINITION: The fibers of π are called **leaves**, or **integral submanifolds** of the distribution B. Globally on M, **a leaf of B** is a maximal connected manifold $Z \hookrightarrow M$ which is immersed to M and tangent to B at each point. A distribution for which Frobenius theorem holds is called **integrable**. If B is integrable, the set of its leaves is called **a foliation**. The leaves are manifolds which are immersed to M, but not necessarily closed.

Frobenius theorem: existence of integral submanifolds

REMARK: To prove the Frobenius theorem for $B \subset TM$, **it suffices to show that each point is contained in an interal submanifold.** In this case, the smooth submersion $U \xrightarrow{\pi} V$ is a projection to the leaf space of the distribution.

REMARK: When B is 1-dimensional (in this case one says that B **has rank 1**, denoted $\operatorname{rk} B = 1$), **Frobenius theorem follows from existence of the diffeomorphism flow associated with a vector field.** Indeed, locally we may assume that B admits a non-degenerate section v. Let $V_t: M \times \mathbb{R} \longrightarrow M$ be the corresponding flow of diffeomorphisms. Then $Z_m := V_t(\{m\} \times \mathbb{R}$ is tangent to v everywhere, hence it is a 1-dimensional manifold immersed in M. Clearly, Z_m is a leaf this distribution. **Since B is a tangent to a foliation, it is integrable.**

Further on we shall need the following exercise.

EXERCISE: Let $V_t = e^{tv}$ be a diffeomorphism flow on M, and $F \subset TM$ a vector bundle. Assume that $[v, F] \subset F$. **Prove that V_t preserves $F \subset TM$.**

Flow of diffeomorphisms

DEFINITION: Let $f: M\times [a,b] \longrightarrow M$ be a smooth map such that for all $t\in [a,b]$ the restriction $f_t := f\big|_{M\times\{t\}} : M \longrightarrow M$ is a diffeomorphism. Then f is called **a flow of diffeomorphisms**.

CLAIM: Let V_t be a flow of diffeomorphisms, $f\in C^\infty M$, and $V_t^*(f)(x) := f(V_t(x))$. Consider the map $\frac{d}{dt}V_t|_{t=c} : C^\infty M \longrightarrow C^\infty M$, with $\frac{d}{dt}V_t|_{t=c}(f) = (V_c^{-1})^*\frac{dV_t}{dt}|_{t=c}f$. **Then $f \longrightarrow (V_t^{-1})^*\frac{d}{dt}V_t^* f$ is a derivation** (that is, a vector field).

Proof: $\frac{d}{dt}V_t^*(fg) = V_t^*(f)\frac{d}{dt}V_t^* g + \frac{d}{dt}V_t^* f V_t^*(g)$ by the Leignitz rule, giving

$$(V_t^{-1})^*\frac{d}{dt}V_t^*(fg) = f(V_t^{-1})^*\frac{d}{dt}V_t^* g + g(V_t^{-1})^*\frac{d}{dt}V_t^* f.$$

■

DEFINITION: The vector field $\frac{d}{dt}V_t|_{t=c}$ is called **the vector field tangent to a flow of diffeomorphisms V_t at $t=c$.**

CLAIM: Let V_t be a flow of diffeomorphisms and X_t the corresponding vector field. **Then for any $\eta\in\Lambda^* M$, one has $\frac{d}{dt}V_t^*(\eta) = \mathrm{Lie}_{X_t}(\eta)$.**

Proof: The operators $\frac{d}{dt}V_t^*$ and Lie_{X_t} are equal on functions, satisfy the Leibitz identity and commute with d. ■

Flow of diffeomorphisms obtained from vector fields

EXERCISE: Let M be a compact manifold, and $\Psi: C^\infty M \longrightarrow C^\infty M$ is a ring automorphism. Prove that **Ψ is induced by an action of a diffeomorphism of M.**

THEOREM: Let M be a compact manifold, and $X_t \in TM$ a family of vector fields smoothly depending on $t \in [0, a]$. **Then there exists a unique diffeomorphism flow V_t, $t \in [0, a]$, such that $V_0 = \mathrm{Id}$ and $\frac{d}{dt} V_t^* = X_t$.**

Proof. Step 1: Given $f \in C^\infty M$, we can solve an equation $\frac{d}{dt} W_t(f) = \mathrm{Lie}_{X_t}(f)$ (here $\mathrm{Lie}_{X_t}(f)$ denotes the derivative along the vector field). The solution $W_t(f)$ exists for all $t \in [0, x]$ and is unique by Peano theorem on existence and uniqueness of solutions of ODE.

Step 2: Since

$$\frac{d}{dt} W_t(fg) = \mathrm{Lie}_{X_t}(f) g + \mathrm{Lie}_{X_t}(g) f = \frac{d}{dt} (W_t(f) W_t(g)),$$

W_t is multiplicative. Also, it is invertible. Applying the previous exercise, we obtain that W_t is a diffeomorphism. ∎

REMARK: If the vector field $X_t = X$ is independent from t, the corresponding diffeomorphism flow **is often denoted as e^{tX}.**

Distributions preserved by a vector field

EXERCISE: Let v, v' be commuting vector fields. **Then the corresponding diffeomorphism flows e^{tv} and $e^{tv'}$ commute.**

CLAIM: Let $V_t = e^{tv}$ be a diffeomorphism flow on M, and $F \subset TM$ a vector bundle. Assume that $[v_t, F] \subset F$. **Then V_t preserves $F \subset TM$.**

Proof. Step 1: Since a non-degenerate vector fields can be linearized, **we can always assume that the vector field v_t is a coordinate vector field, $v_t = x_1$.** Since the statement is local, we can always assume that M is an open subset in $\mathbb{R}^n$ with coordinates $x_1, ..., x_n$.

Step 2: Let $R_1, ..., R_r$ be a basis in F. We write $R_i = \sum f_{ij} \frac{d}{dx_j}$; then $[v_t, R_i] = \sum \frac{df_{ij}}{dx_i} \frac{d}{dx_j}$. In these terms $[v_t, F] \subset F$ is written as $\frac{dR_i}{dx_1} = \sum a_{ij} R_j$, where a_{ij} are appropriate functions on M.

Distributions preserved by a vector field (2)

CLAIM: Let $V_t = e^{tv}$ be a diffeomorphism flow on M, and $F \subset TM$ a vector bundle. Assume that $[v_t, F] \subset F$. **Then V_t preserves $F \subset TM$.**

Step 2: Let $R_1, ..., R_r$ be a basis in F. We write $R_i = \sum f_{ij} \frac{d}{dx_j}$; then $[v_t, R_i] = \sum \frac{df_{ij}}{dx_1} \frac{d}{dx_j}$. In these terms $[v_t, F] \subset F$ is written as $\frac{dR_i}{dx_1} = \sum a_{ij} R_j$, where a_{ij} are appropriate functions on M.

Step 3: Let $F_i(t) := V_t(R_i) \in TM$. By definition, $V_t = e^{td/dx_1}$, hence $\frac{d}{dt} V_t(R_i) = [d/dx_1, V_t(R_i)]$. Then, $F_i(t)$ is a solution of a vector-valued differential equation

$$\frac{d}{dt} F_i(t) = [d/dx_1, F_i(t)], \qquad (*)$$

with initial values $F_i(t) = R_i$. The solution can be found as $F_i(t) = \sum_{j=1}^r b_{ij}(t) R_j$ because

$$\left[d/dx_1, \sum_{j=1}^r b_{ij}(t) R_j \right] = \sum_{j=1}^r b_{ij}(t) \sum_{k=1}^r a_{jk} R_k + \frac{b_{ij}}{dx_1} R_j.$$

Then $F_i(t) = \sum_{j=1}^r b_{ij}(t) R_j$ is a solution of (*) if for all $i, j = 1, ..., r$, we have

$$\frac{db_{ij}(t)}{dt} = \frac{db_{ij}}{dx_1} + \sum_{i=1}^r \sum_{k=1}^r b_{ik} a_{kj}.$$

This differential equation has a unique solution with a given initial value. ■

Basic sub-bundles (1)

DEFINITION: Let $B \subset TM$ be an involutive sub-bundle. A sub-bundle $F \subset TM$ is called **basic** for B if $F \supset B$ and for all $b \in B, b' \in F$, one has $[b, b'] \in F$.

LEMMA: Let $B \subset TM$ be an integrable distribution, $\pi: M \longrightarrow M_1$ projection to the leaf space of B, and $F \supset B$ a sub-bundle of TM containing B. Then the following conditions are equivalent: **(a) F is basic for B.**
(b) There exists a sub-bundle $F_1 \subset TM_1$ such that $\pi^{-1}(F_1) = F$.

Proof: Next slide.

Basic sub-bundles (2)

LEMMA: Let $B \subset TM$ be an integrable distribution, $\pi : M \longrightarrow M_1$ projection to the leaf space of B, and $F \supset B$ a sub-bundle of TM containing B. Then the following conditions are equivalent: **(a) F is basic for B.**
(b) There exists a sub-bundle $F_1 \subset TM_1$ such that $\pi^{-1}F_1 = F$.

Proof. Step 1: Consider coordinates $x_1, ..., x_n$ on M such that $x_{k+1} = \pi^*(x'_{k+1}, ..., x_n = \pi^*(x_n)$, where $x'_i, i = k+1, k+2, ..., n$ are coordinates on M_1, and $\frac{d}{dx_1}, ..., \frac{d}{dx_k}$ generate B. Locally such coordinates always exist, because B is integrable. Denote by G a subgroup of $\operatorname{Diff}(M)$ obtained by exponents of $\frac{d}{dx_1}, ..., \frac{d}{dx_k}$. Since $[B, F] \subset F$, the corresponding diffeomorphisms preserve F. **Therefore, F is a G-invariant sub-bundle of TM.**

Step 2: Any G-invariant sub-bundle $F \supset B$ is obtained as $\pi^{-1}(F_1)$ for some sub-bundle $F_1 \subset TM_1 = M/G$. Indeed, since the action of G_1 is free, the bundle F is generated over $C^\infty M$ by G-invariant sections. However, any G-invariant bundle F containing B is generated by G-invariant sections, which can be lifted from M/G **(check this)**.

Step 3: Conversely, if F is lifted from $M_1 = M/G$, it is G-invariant, hence $e^{tb}(b') \subset F$, and this gives $[b, b'] \subset F$ **(check this)**. ■

Frobenius theorem (proof)

Frobenius Theorem: Let $B \subset TM$ be a sub-bundle. Then B is involutive if and only if each point $x \in M$ has a neighbourhood $U \ni x$ and **a smooth submersion $U \xrightarrow{\pi} V$ such that B is its vertical tangent space: $B = T_\pi M$.**

Proof. Step 1: Consider a rank 1 sub-bundle $B_1 \subset B$. Using the diffeomorphism flow as above, we prove that B_1 is integrable. Since $[B_1, B] \subset B$, the bundle B is basic with respect to B_1. **Therefore, $B = \pi^{-1}(B')$ for some $B' \subset TM_1$, where M_1 is the leaf space of B_1.**

Step 2: Let $\pi : M \longrightarrow M_1$ be the projection to the leaf space. Then $B = \pi^{-1}(B')$, where $\operatorname{rk} B' = \operatorname{rk} B - 1$. Using induction in $\operatorname{rk} B$, we can assume that B' is integrable. Let $\pi_0 : M_1 \longrightarrow M_0$ be the projection to the leaf space of B', defined locally in M. **Then $\pi \circ \pi_0 : M \longrightarrow M_0$ is the projection to the leaf space of B.** ■

Complex manifolds (reminder)

DEFINITION: A holomorphic function on $\mathbb{C}^n$ is a function $f: \mathbb{C}^n \longrightarrow \mathbb{C}$ such that df is complex linear, that is $df \in \Lambda^{1,0}(M)$.

REMARK: Holomorphic functions form a sheaf.

DEFINITION: A complex manifold M is a ringed space which is locally isomorphic to an open ball in $\mathbb{C}^n$ with a sheaf of holomorphic functions.

REMARK: In other words, M **is covered with open balls embedded to** $\mathbb{C}^n$ and transition functions (being coordinate functions for one ball considered in other coordinate system) **are holomorphic.**

Integrability of almost complex structures (reminder)

DEFINITION: An almost complex structure I on a manifold is called **integrable** if any point of M has a neighbourhood U diffeomorphic to an open subset of $\mathbb{C}^n$, in such a way that the almost complex structure I is induced by the standard one on $U \subset \mathbb{C}^n$.

CLAIM: Complex structure on a manifold M uniquely determines an integrable almost complex structure, and is determined by it.

Proof: Complex structure on a manifold M is determined by the sheaf of holomorphic functions $\mathcal{O}_M$, and $\mathcal{O}_M$ is determined by I as explained above. Therefore, an integrable almost complex structure defines a complex structure. Conversely, every complex structure gives a sub-bundle in $\Lambda^{1,0}(M) = d\mathcal{O}_M \subset \Lambda^1(M, \mathbb{C})$, and **such a sub-bundle defines an almost complex structure by Remark 1.** ■

Frobenius form (reminder)

CLAIM: Let $B \subset TM$ be a sub-bundle of a tangent bundle of a smooth manifold. Given vector fiels $X, Y \in B$, consider their commutator $[X, Y]$, and lets $\Psi(X, Y) \in TM/B$ be the projection of $[X, Y]$ to TM/B. **Then $\Psi(X, Y)$ is $C^\infty(M)$-linear in X, Y:**

$$\Psi(fX, Y) = \Psi(X, fY) = f\Psi(X, Y).$$

Proof: Leibnitz identity gives $[X, fY] = f[X, Y] + X(f)Y$, and the second term belongs to B, hence does not influence the projection to TM/B. ■

DEFINITION: This form is called **the Frobenius form** of the sub-bundle $B \subset TM$. This bundle is called **involutive**, or **integrable**, or **holonomic** if $\Psi = 0$.

EXERCISE: **Give an example of a non-integrable sub-bundle.**

Formal integrability (reminder)

DEFINITION: An almost complex structure I on (M, I) is called **formally integrable** if $[T^{1,0}M, T^{1,0}] \subset T^{1,0}$, that is, if $T^{1,0}M$ is involutive.

DEFINITION: The Frobenius form $\Psi \in \Lambda^2(\Lambda^{1,0}M) \otimes T^{0,1}M$ is called **the Nijenhuis tensor**.

CLAIM: If a complex structure I on M is integrable, it is formally integrable.

Proof: Locally, the bundle $T^{1,0}(M)$ is generated by d/dz_i, where z_i are complex coordinates. These vector fields commute, hence satisfy $[d/dz_i, d/dz_j] \in T^{1,0}(M)$. This means that the Frobenius form vanishes. ■

THEOREM: (Newlander-Nirenberg)
A complex structure I on M is integrable if and only if it is formally integrable.

Proof: (real analytic case) next lecture, probably.

REMARK: In dimension 1, formal integrability is automatic. Indeed, $T^{1,0}M$ is 1-dimensional, hence all skew-symmetric 2-forms on $T^{1,0}M$ vanish.

Real analytic manifolds

DEFINITION: A real analytic function on an open set $U \subset \mathbb{R}^n$ is a function which admits a Taylor expansion near each point $x \in U$:

$$f(z_1 + t_1, z_2 + t_2, ..., z_n + t_n) = \sum_{i_1, ..., i_n} a_{i_1, ..., i_n} t_1^{i_1} t_2^{i_2} ... t_n^{i_n}.$$

(here we assume that the real numbers t_i satisfy $|t_i| < \varepsilon$, where ε depends on f and M).

REMARK: Clearly, **real analytic functions constitute a sheaf.**

DEFINITION: A **real analytic manifold** is a ringed space which is locally isomorphic to an open ball $B \subset \mathbb{R}^n$ with the sheaf of of real analytic functions.

Involutions

DEFINITION: An involution is a map $\iota : M \longrightarrow M$ such that $\iota^2 = \mathrm{Id}_M$.

EXERCISE: Prove that **any linear involution on a real vector space V is diagonalizable,** with eigenvalues ± 1.

Theorem 1: Let M be a smooth manifold, and $\iota : M \longrightarrow M$ an involutiin. **Then the fixed point set N of ι is a smooth submanifold.**

Proof. Step 1: Inverse function theorem. Let $m \in M$ be a point on a smooth k-dimensional manifold and $f_1, ..., f_k$ functions on M such that their differentials $df_1, ..., df_k$ are linearly independent in m. Then $f_1, ..., f_k$ **define a coordinate system in a neighbourhood of a, giving a diffeomorphism of this neighbourhood to an open ball.**

Step 2: Assume that $d\iota$ has k eigenvalues 1 on $T_m M$, and $n-k$ eigenvalues -1. Choose a coordinate system $x_1, ..., x_n$ on M around a point $m \in N$ such that $dx_1|_m, ..., dx_k|_m$ are ι-invariant and $dx_{k+1}|_m, ..., dx_n|_m$ are ι-anti-invariant. Let $y_1 = x_1 + \iota^* x_1$, $y_2 = x_2 + \iota^* x_2$, ... $y_k = x_k + \iota^* x_k$, and $y_{k+1} = x_{k+1} - \iota^* x_{k+1}$, $y_{k+2} = x_{k+2} - \iota^* x_{k+2}$, ... $y_n = x_n - \iota^* x_n$. Since $dx_i|_m = xy_i|_m$, these differentials are linearly independent in m. By Step 1, **functions y_i define an ι-invariant coordinate system on an open neighbourhood of m, with N given by equations $y_{k+1} = y_{k+2} = ... = y_n = 0$.** ■

Real structures

DEFINITION: **An involution** is a map $\iota : M \longrightarrow M$ such that $\iota^2 = \mathrm{Id}_M$. **A real structure** on a complex vector space $V = \mathbb{C}^n$ is an $\mathbb{R}$-linear involution $\iota : V \longrightarrow V$ such that $\iota(\lambda x) = \overline{\lambda}\iota(x)$ for any $\lambda \in \mathbb{C}$.

DEFINITION: A map $\Psi : M \longrightarrow M$ on an almost complex manifold (M, I) is called **antiholomorphic** if $d\Psi(I) = -I$. A function f is called **antiholomorphic** if $\overline{f}$ is holomorphic.

EXERCISE: Prove that **antiholomorphic function on M defines an antiholomorphic map from M to $\mathbb{C}$.**

EXERCISE: Let ι be a smooth map from a complex manifold M to itself. Prove that ι **is antiholomorphic if and only if $\iota^*(f)$ is antiholomorphic for any holomorphic function f on $U \subset M$.**

DEFINITION: **A real structure** on a complex manifold M is an antiholomorphic involution $\tau : M \longrightarrow M$.

EXAMPLE: Complex conjugation defines a real structure on $\mathbb{C}^n$.

Real analytic manifolds and real structures

PROPOSITION: Let $M_{\mathbb{R}} \subset M_{\mathbb{C}}$ be a fixed point set of an antiholomorphic involution ι, U_i a complex analytic atlas, and $\Psi_{ij} : U_{ij} \longrightarrow U_{ij}$ the gluing functions. **Then, for appropriate choice of coordinate systems all Ψ_{ij} are real analytic on $M_{\mathbb{R}}$, and define a real analytic atlas on the manifold $M_{\mathbb{R}}$.**

Proof. Step 1: Let $z_1, ..., z_n$ be a holomorphic coordinate system on $M_{\mathbb{C}}$ in a neighbourhood of $m \in M_{\mathbb{R}}$ such that $\iota(dz_i) = d\overline{z}_i$ in $T^*_m M$. Such a coordinate system can be chosen by taking linear functions with prescribed differentials in m. **Replacing z_i by $x_i := z_i + \iota^*(\overline{z}_i)$, we obtain another coordinate system x_i on M** (compare with Theorem 1).

Step 2: This new coordinate system satisfies $\iota^* x_i = \overline{x}_i$, hence $M_{\mathbb{R}}$ in these coordinates is giving by equation $\operatorname{im} x_1 = \operatorname{im} x_2 = ... = \operatorname{im} x_n = 0$. **All gluing functions from such coordinate system to another one of this type satisfy $\Psi_{ij}(\overline{z}_i) = \overline{\Psi_{ij}(\overline{z}_i)}$, hence they are real on $M_{\mathbb{R}}$.** ■

Real analytic manifolds and real structures (2)

PROPOSITION: Any real analytic manifold can be obtained from this construction.

Proof. Step 1: Let $\{U_i\}$ be a locally finite atlas of a real analytic manifold M, and $\Psi_{ij} : U_{ij} \longrightarrow U_{ij}$ the gluing map. We realize U_i as an open ball with compact closure in $\operatorname{Re}(\mathbb{C}^n) = \mathbb{R}^n$. By local finiteness, there are only finitely many such Ψ_{ij} for any given U_i. Denote by B_ε an open ball of radius ε in the n-dimensional real space $\operatorname{im}(\mathbb{C}^n)$.

Step 2: Let $\varepsilon > 0$ be a sufficiently small real number such that all Ψ_{ij} can be extended to gluing functions $\tilde{\Psi}_{ij}$ on the open sets $\tilde{U}_i := U_i \times B_\varepsilon \subset \mathbb{C}^n$. **Then $(\tilde{U}_i, \Psi_{ij})$ is an atlas for a complex manifold $M_\mathbb{C}$.** Since all Ψ_{ij} are real, they are preserved by natural involution acting on B_ε as -1 and on U_i as identity. This involution defines a real structure on $M_\mathbb{C}$. Clearly, M is the set of its fixed points. ■

Complexification

DEFINITION: Let $M_{\mathbb{R}}$ be a real analytic manifold, and $M_{\mathbb{C}}$ a complex analytic manifold equipped with an antiholomorphic involution, such that $M_{\mathbb{R}}$ is the set of its fixed points. Then $M_{\mathbb{C}}$ is called **complexification** of $M_{\mathbb{R}}$.

DEFINITION: A tensor on a real analytic manifold is called **real analytic** if it is expressed locally by a sum of coordinate monomials with real analytic coefficients.

CLAIM: Let $M_{\mathbb{R}}$ be a real analytic manifold, $M_{\mathbb{C}}$ its complexification, and Φ a tensor on $M_{\mathbb{R}}$. **Then Φ is real analytic if and only if Φ can be extended to a holomorpic tensor $\Phi_{\mathbb{C}}$ in some neighbourhood of $M_{\mathbb{R}}$ inside $M_{\mathbb{C}}$.**

Proof: The "if" part is clear, because every complex analytic tensor on $M_{\mathbb{C}}$ is by definition real analytic on $M_{\mathbb{R}}$.

Conversely, suppose that Φ is expressed by a sum of coordinate monomials with real analytic coefficients f_i. Let $\{U_i\}$ be a cover of M, and $\tilde{U}_i := U_i \times B_\varepsilon$ the corresponding cover of a neighbourhood of $M_{\mathbb{R}}$ in $M_{\mathbb{C}}$ constructed above. Chosing ε sufficiently small, we can assume that the Taylor series giving coefficients of Φ converges on each $\tilde{U}_i$. **We define $\Phi_{\mathbb{C}}$ as the sum of these series.** ■

Extension of tensors to a complexification

Lemma 1: Let X be an open ball in $\mathbb{C}^n$ equipped with the standard anticomplex involution, $X_{\mathbb{R}} = X \cap \mathbb{R}^n$ its fixed point set, and α a holomorphic tensor on X vanishing in $X_{\mathbb{R}}$. **Then $\alpha = 0$.**

Proof: Any holomorphic function which vanishes on $\mathbb{R}^n$ has all its derivatives is equal zero. Therefore its Taylor series vanish. Such a function vanishes on $\mathbb{C}^n$ by analytic continuation principle. This argument can be applied to all coefficients of α. ■

DEFINITION: An almost complex structure I on a real analytic manifold is **real analytic** if I is a real anaytic tensor.

COROLLARY: Let (M, I) be a real analytic almost complex manifold, $M_{\mathbb{C}}$ its complexification, and $I_{\mathbb{C}} : TM_{\mathbb{C}} \longrightarrow TM_{\mathbb{C}}$ the holomorphic extension of I to $M_{\mathbb{C}}$. **Then $I_{\mathbb{C}}^2 = -\operatorname{Id}$.**

Proof: The tensor $I_{\mathbb{C}}^2 + \operatorname{Id}$ is holomorphic and vanishes on $M_{\mathbb{R}}$, hence the previous lemma can be applied. ■

Underlying real analytic manifold

REMARK: A complex analytic map $\Phi: \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is real analytic as a map $\mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$. Indeed, the coefficients of Φ are real and imaginary parts of holomorphic functions, and real and imaginary parts of holomorphic functions can be expressed as Taylor series of the real variables.

DEFINITION: Let M be a complex manifold. The **underlying real analytic manifold** is the same manifold, with the same gluing functions, considered as real analytic maps.

DEFINITION: Let M be a complex manifold. The **complex conjugate manifold** is the same manifold with almost complex structure $-I$ and anti-holomorphic functions on M holomorphic on $\overline{M}$.

CLAIM: Let M be an integrable almost complex manifold. Denote by $M_{\mathbb{R}}$ its underlying real analytic manifold. **Then a complexification of $M_{\mathbb{R}}$ can be given as $M_{\mathbb{C}} := M \times \overline{M}$,** with the anticomplex involution $\tau(x, y) = (y, x)$.

Proof: Clearly, the fixed point set of τ is the diagonal, identified with $M_{\mathbb{R}} = M$ as usual. Both holomorphic and antiholomorphic functions on $M_{\mathbb{R}}$ are obtained as restrictions of holomorphic functions from $M_{\mathbb{C}}$, hence the sheaf of real analytic functions on $M_{\mathbb{R}}$ is a real part of the sheaf $\mathcal{O}_{M_{\mathbb{C}}}$ of holomorphic functions on $M_{\mathbb{C}}$. ■

Holomorphic and antiholomorphic foliations

DEFINITION: Let $B \subset TM$ be a sub-bundle. The **foliation associated with B** is a family of submanifolds $X_t \subset U$, defined for each sufficiently small subset of M, called **the leaves of the foliation**, such that B is the bundle of vectors tangent to X_t. In this case, X_t are called **the leaves** of the foliation.

REMARK: The famous "Frobenius theorem" says that B **is involutive if and only if it is tangent to a foliation.**

REMARK: Let (M,I) be a real analytic almost complex manifold, and $M_{\mathbb C}$ its complexification. Replacing $M_{\mathbb C}$ by a smaller neighbourhood of M, we may assume that the tensor I is extended to an endomorphism $I: TM_{\mathbb C} \longrightarrow TM_{\mathbb C}$, $I^2 = -\operatorname{Id}$. **Since $TM_{\mathbb C}$ is a complex vector bundle, I acts there with the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, giving a decomposition $TM_{\mathbb C} = T^{1,0}M_{\mathbb C} \oplus T^{0,1}M_{\mathbb C}$**

DEFINITION: Holomorphic foliation is a foliation tangent to $T^{1,0}M_{\mathbb C}$, **antiholomorphic foliation** is a foliation tangent to $T^{0,1}M_{\mathbb C}$.

Antiholomorphic foliation on $M_{\mathbb C} = M \times \overline{M}$.

CLAIM: Let (M,I) be a integrable almost complex manifold, $M_{\mathbb C} = M \times \overline{M}$ its complexification, and $\pi, \overline{\pi}$ projections of $M_{\mathbb C}$ to M and $\overline{M}$. **Then the fibers of $\overline{\pi}$ is a holomorphic foliation, and the fibers of π is a holomorphic foliation.**

Proof: Let $TM_{\mathbb C} = T' \oplus T''$ be a decomposition of $TM_{\mathbb C}$ onto part tangent to fibers of $\overline{\pi}$ and tangent to fibers of π. **On $M_{\mathbb R}$ the decomposition $TM_{\mathbb C} = T' \oplus T''$ coincides with the decomposition $TM \otimes \mathbb C = T^{1,0}M \oplus T^{0,1}M$.** By Lemma 1 the same is true everywhere on $M_{\mathbb C}$. ■

COROLLARY: Let (M,I) be a integrable almost complex manifold. **Then I is a real analytic almost complex structure.**

Proof: It was extended to $M_{\mathbb C}$ in the previous claim. ■

Corollary 1: Let (M,I) be a real analytic almost complex manifold. Then holomorphic functions on $M_{\mathbb C}$ which are constant on the leaves of antiholomoirphic foliation **restrict to holomorphic functions on $(M,I) \subset M_{\mathbb C}$.**

Proof: Such functions are constant in the $(0,1)$-direction on $TM \otimes \mathbb C$. ■

Integrability of real analytic almost complex structure

THEOREM: ("linearization of a vector field") Let $v \in TM$ be a nowhere vanishing vector field on M. **Then there exists a family of 1-dimensional submanifolds passing through each point of M such that v is tangent to these submanifolds at each point of M.**

THEOREM: Let (M, I) be a real analytic almost complex manifold, $\dim_{\mathbb{R}} M = 2$. **Then M is integrable.**

Proof. Step 1: Consider the complexification $M_{\mathbb{C}}$ of M, and let $TM_{\mathbb{C}} = T^{1,0}M_{\mathbb{C}} \oplus T^{0,1}M_{\mathbb{C}}$ be the decomposition defined above. By "linearization of a vector field" theorem, there exists a foliation tangent to $T^{0,1}M_{\mathbb{C}}$ and one tangent to $T^{1,0}M_{\mathbb{C}}$. Since the leaves of these foliations are transversal, **locally $M_{\mathbb{C}}$ is a product of M' and M'' which are identified with the space of leaves of $T^{0,1}M_{\mathbb{C}}$ and $T^{1,0}M_{\mathbb{C}}$.**

Step 2: Locally, functions on M' can be lifted to $M' \times M'' = M_{\mathbb{C}}$, giving functions which are constant on the leaves of the foliation tangent to $T^{0,1}M_{\mathbb{C}}$. By Corollary 1, such functions are holomorphic on (M, I). Choose a collection of $n = \frac{1}{2}\dim_{\mathbb{R}} M$ holomorphic functions $f_1, ...f_n$ on $M_{\mathbb{C}}$ which are constant on the leaves of $T^{0,1}M_{\mathbb{C}}$ and have linearly independent differentials in $x \in M \subset M_{\mathbb{C}}$. By inverse function theorem, $f_1, ..., f_n$ **holomorphic coordinate system in a neigbourhood of $x \in (M, I)$,** and the transition functions between such coordinate systems are by construction holomorphic. ■