Complex geometry

lecture 4: Another proof of Frobenius theorem; real analytic manifolds

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HSE, room 306, 16:20,

October 3, 2020

Distributions (reminder)

DEFINITION: Distribution on a manifold is a sub-bundle $B \subset TM$

REMARK: Let $\Pi: TM \longrightarrow TM/B$ be the projection, and $x, y \in B$ some vector fields. Then $[fx, y] = f[x, y] - D_y(f)x$. This implies that **$\Pi([x, y])$ is $C^\infty(M)$-linear as a function of x and y.**

DEFINITION: The map $[B, B] \longrightarrow TM/B$ we have constructed is called **Frobenius bracket** (or **Frobenius form**); it is a skew-symmetric $C^\infty(M)$-linear form on B with values in TM/B.

DEFINITION: A distribution is called **integrable**, or **holonomic**, or **involutive**, if its Frobenius form vanishes.

Smooth submersions (reminder)

DEFINITION: Let $\pi: M \longrightarrow M'$ be a smooth map of manifolds. This map is called **submersion** if at each point of M the differential $D\pi$ is surjective, and **immersion** if it is injective.

CLAIM: Let $\pi: M \longrightarrow M'$ be a submersion. Then each $m \in M$ has a neighbourhood $U \cong V \times W$, where V, W are smooth and **$\pi|_U$ is a projection of $V \times W = U \subset M$ to $W \subset M'$ along V.**

EXERCISE: Deduce this result from the inverse function theorem.

EXERCISE: ("Ehresmann's fibration theorem")
Let $\pi: M \longrightarrow M'$ be a smooth submersion of compact manifolds. Prove that π is a locally trivial fibration.

DEFINITION: Vertical tangent space $T_\pi M \subset TM$ of a submersion $\pi: M \longrightarrow M'$ is the kernel of $D\pi$.

CLAIM: Let $\pi: M \longrightarrow M'$ be a submersion and $T_\pi M \subset TM$ the vertical tangent space. **Then $T_\pi M$ is an involutive subbundle.**

Proof: $D_\pi([X,Y]) = [D_\pi(X), D_\pi(Y)] = 0$ for any $X, Y \in \ker D_\pi$. ∎

Frobenius theorem (statement)

Frobenius Theorem: Let $B \subset TM$ be a sub-bundle. Then B is involutive if and only if each point $x \in M$ has a neighbourhood $U \ni x$ and **a smooth submersion $U \xrightarrow{\pi} V$ such that B is its vertical tangent space: $B = T_\pi M$.**

REMARK: The implication **"$B = T_\pi M$" $\Rightarrow$ "Frobenius form vanishes"** was proven above.

DEFINITION: The fibers of π are called **leaves**, or **integral submanifolds** of the distribution B. Globally on M, **a leaf of B** is a maximal connected manifold $Z \hookrightarrow M$ which is immersed to M and tangent to B at each point. A distribution for which Frobenius theorem holds is called **integrable**. If B is integrable, the set of its leaves is called **a foliation**. The leaves are manifolds which are immersed to M, but not necessarily closed.

Proof of Frobenius Theorem: Lie group action

DEFINITION: A **Lie group** is a smooth manifold equipped with a group structure in such a way that the group operations are smooth. **An action** of a Lie group G on a manifold M is a smooth map $G \times M \longrightarrow M$ inducing the group action.

DEFINITION: The Lie algebra of a Lie group is an algebra of left-invariant vector fields. Since the left action of G on itself is free and transitive, **the Lie algebra of G is in bijective correspondence with T_eG.**

REMARK: Let $\rho: G \times M \longrightarrow M$ be a Lie group action on a manifold M. Then $d\rho$ induces a map from the Lie algebra of G to the Lie algebra of the manifold M. **The corresponding vector fields are tangent to the orbits of G.**

Claim 1: Suppose that G is a Lie group acting on a manifold M. Let $B \subset TM$ be sub-bundle generated by the vector fields from Lie algebra of G. Then B **is integrable,** that is, Frobenius theorem holds of $B \subset TM$.

Proof: The orbits of the G-action are tangent to $B \subset TM$. ■

Proof of Frobenius Theorem: preliminaries

Exercise 1: Let u, v be commuting vector fields on a manifold M, and e^{tu}, e^{tv} be corresponding diffeomorphism flows. **Prove that e^{tu}, e^{tv} commute.**

Remark 1: Let $\pi: M \longrightarrow M_1$ be a smooth submersion, and $v \in TM$ a vector fields which satisfies

$$d\pi(v)|_x = d\pi(v)|_y \qquad (*)$$

for any $x, y \in \pi^{-1}(z)$ and any $z \in M_1$. **In this case, the vector field $d\pi(v)$ is well defined on M.**

Exercise 2: Let $\pi: M \longrightarrow M_1$ be a smooth submersion, and $u, v \in TM$ vector fields which satisfy (*). Consider the vector fields $u_1 := d\pi(u)$ and $v_1 := d\pi(v) \in TM_1$ defined as in Remark 1. **Prove that the commutator $[u, v]$ satisfies (*) and, moreover, $[u_1, v_1] = d\pi([u, v])$.**

Proof of Frobenius Theorem: commuting vector fields

Exercise 1: Let u, v be commuting vector fields on a manifold M, and e^{tu}, e^{tv} be corresponding diffeomorphism flows. **Prove that e^{tu}, e^{tv} commute.**

Solution. Step 1: The statement is local, and trivial in any open set where $u = 0$, hence it suffices to prove it in a coordinate chart where u is non-degenerate. Since all non-degenerate vector fields can be linearized, **we can always assume that the vector field u is a coordinate vector field, $u = d/dx_1$.** Then $e^{tu}(x_1, ..., x_n) = (x_1 + t, ..., x_n)$.

Step 2: For any vector field $v = \sum_i a_i d/dx_i$, one has $\left[u, \sum_i a_i \frac{d}{dx_i}\right] = \sum \frac{da_i}{dx_1} \frac{d}{dx_i}$. Therefore, $[u, v] = 0$ **is equivalent to the coefficients a_i being constant in** x_1. This implies that the parallel transport along x_1 preserves v. Therefore, it also preserves e^{tv}, and the corresponding diffeomorphisms commute. ■

Frobenius theorem (proof)

Frobenius Theorem: Let $B \subset TM$ be a sub-bundle. Then B is involutive if and only if each point $m \in M$ has a neighbourhood $U \ni m$ and **a smooth submersion $U \xrightarrow{\pi} V$ such that B is its vertical tangent space: $B = T_\pi M$.**

Proof. Step 1: The "if" part is clear. The statement of Frobenius Theorem is local, hence we may replace M be a small neighbourhood of a given point. We are going to show that B locally has a basis of commuting vector fields. By Exercise 1, **these vector fields can be locally integrated to a commutative group action,** and Frobenius Theorem follows from Claim 1.

Step 2: Consider a smooth submersion $M \longrightarrow M_1$ inducing an isomorphism from B to TM_1. Let $\zeta_1, ..., \zeta_k$ be the coordinate vector fields on M_1. Since $d\sigma : \ B|_x \longrightarrow T_{\sigma(x)}M_1$ is an isomorphism, there exist unique vector fields $\xi_1, ..., \xi_k \in B$ such that $d\sigma(\xi_i) = \zeta_i$. By Exercise 2, $d\sigma([\xi_1, \xi_j]) = [\zeta_i, \zeta_j] = 0$. However, $[\xi_1, \xi_j]$ is a section of B, and $d\sigma : \ B|_m \longrightarrow T_{\sigma(m)}M_1$ is an isomorphism, hence $d\sigma([\xi_1, \xi_j]) = 0$ implies $[\xi_1, \xi_j] = 0$. We have shown that **B admits a basis of commuting vector fiels.** ■

Complex manifolds (reminder)

DEFINITION: **A holomorphic function** on $\mathbb{C}^n$ is a function $f: \mathbb{C}^n \longrightarrow \mathbb{C}$ such that df is complex linear, that is $df \in \Lambda^{1,0}(M)$.

REMARK: Holomorphic functions form a sheaf.

DEFINITION: **A complex manifold** M is a ringed space which is locally isomorphic to an open ball in $\mathbb{C}^n$ with a sheaf of holomorphic functions.

REMARK: In other words, M **is covered with open balls embedded to** $\mathbb{C}^n$ and transition functions (being coordinate functions for one ball considered in other coordinate system) **are holomorphic.**

Integrability of almost complex structures (reminder)

DEFINITION: An almost complex structure I on a manifold is called **integrable** if any point of M has a neighbourhood U diffeomorphic to an open subset of $\mathbb{C}^n$, in such a way that the almost complex structure I is induced by the standard one on $U \subset \mathbb{C}^n$.

CLAIM: Complex structure on a manifold M uniquely determines an integrable almost complex structure, and is determined by it.

Proof: Complex structure on a manifold M is determined by the sheaf of holomorphic functions $\mathcal{O}_M$, and $\mathcal{O}_M$ is determined by I as explained above. Therefore, an integrable almost complex structure defines a complex structure. Conversely, every complex structure gives a sub-bundle in $\Lambda^{1,0}(M) = d\mathcal{O}_M \subset \Lambda^1(M, \mathbb{C})$, and **such a sub-bundle defines an almost complex structure by Remark 1.** ∎

Formal integrability (reminder)

DEFINITION: An almost complex structure I on (M, I) is called **formally integrable** if $[T^{1,0}M, T^{1,0}] \subset T^{1,0}$, that is, if $T^{1,0}M$ is involutive.

DEFINITION: The Frobenius form $\Psi \in \Lambda^2(\Lambda^{1,0}M) \otimes T^{0,1}M$ is called **the Nijenhuis tensor**.

CLAIM: If a complex structure I on M is integrable, it is formally integrable.

Proof: Locally, the bundle $T^{1,0}(M)$ is generated by d/dz_i, where z_i are complex coordinates. These vector fields commute, hence satisfy $[d/dz_i, d/dz_j] \in T^{1,0}(M)$. This means that the Frobenius form vanishes. ■

THEOREM: (Newlander-Nirenberg)
A complex structure I on M is integrable if and only if it is formally integrable.

Proof: (real analytic case) next lecture, probably.

REMARK: In dimension 1, formal integrability is automatic. Indeed, $T^{1,0}M$ is 1-dimensional, hence all skew-symmetric 2-forms on $T^{1,0}M$ vanish.

Real analytic manifolds

DEFINITION: A real analytic function on an open set $U \subset \mathbb{R}^n$ is a function which admits a Taylor expansion near each point $x \in U$:

$$f(z_1 + t_1, z_2 + t_2, ..., z_n + t_n) = \sum_{i_1,...,i_n} a_{i_1,...,i_n} t_1^{i_1} t_2^{i_2} ... t_n^{i_n}.$$

(here we assume that the real numbers t_i satisfy $|t_i| < \varepsilon$, where ε depends on f and M).

REMARK: Clearly, **real analytic functions constitute a sheaf.**

DEFINITION: A **real analytic manifold** is a ringed space which is locally isomorphic to an open ball $B \subset \mathbb{R}^n$ with the sheaf of of real analytic functions.

Involutions

DEFINITION: An involution is a map $\iota: M \longrightarrow M$ such that $\iota^2 = \operatorname{Id}_M$.

EXERCISE: Prove that **any linear involution on a real vector space V is diagonalizable,** with eigenvalues ± 1.

Theorem 1: Let M be a smooth manifold, and $\iota: M \longrightarrow M$ an involutiin. **Then the fixed point set N of ι is a smooth submanifold.**

Proof. Step 1: Inverse function theorem. Let $m \in M$ be a point on a smooth k-dimensional manifold and $f_1, ..., f_k$ functions on M such that their differentials $df_1, ..., df_k$ are linearly independent in m. Then $f_1, ..., f_k$ **define a coordinate system in a neighbourhood of a, giving a diffeomorphism of this neighbourhood to an open ball.**

Step 2: Assume that $d\iota$ has k eigenvalues 1 on $T_m M$, and $n-k$ eigenvalues -1. Choose a coordinate system $x_1, ..., x_n$ on M around a point $m \in N$ such that $dx_1|_m, ..., dx_k|_m$ are ι-invariant and $dx_{k+1}|_m, ..., dx_n|_m$ are ι-anti-invariant. Let $y_1 = x_1 + \iota^* x_1$, $y_2 = x_2 + \iota^* x_2$, ... $y_k = x_k + \iota^* x_k$, and $y_{k+1} = x_{k+1} - \iota^* x_{k+1}$, $y_{k+2} = x_{k+2} - \iota^* x_{k+2}$, ... $y_n = x_n - \iota^* x_n$. Since $dx_i|_m = dy_i|_m$, these differentials are linearly independent in m. By Step 1, **functions y_i define an ι-invariant coordinate system on an open neighbourhood of m, with N given by equations $y_{k+1} = y_{k+2} = ... = y_n = 0$.** ■

Real structures on complex vector spaces

DEFINITION: An involution is a map $\iota: M \longrightarrow M$ such that $\iota^2 = \mathrm{Id}_M$. **A real structure** on a complex vector space $V = \mathbb{C}^n$ is an $\mathbb{R}$-linear involution $\iota: V \longrightarrow V$ such that $\iota(\lambda x) = \overline{\lambda}\iota(x)$ for any $\lambda \in \mathbb{C}$.

CLAIM: Let ι be a real structure on a complex vector space V, and $V^\iota \subset V$ the space of V-invariant vectors. **Then $\dim_{\mathbb{R}} V^\iota = \dim_{\mathbb{C}} V$, and $V = V^\iota \otimes_{\mathbb{R}} \mathbb{C}$.**

Proof. Step 1: Let $x_1, ..., x_n$ be a basis in V^ι, and $\sum_i \alpha_i x_i = 0$ a linear relation in V, with $\alpha_i \in \mathbb{C}$. Then $0 = \iota\left(\sum_i \alpha_i x_i\right) = \sum_i \overline{\alpha}_i x_i$. Averaging these two relations, we obtain $\sum \mathrm{Re}\,\alpha_i x_i = 0$. Since x_i are linearly independent over $\mathbb{R}$, this implies $\mathrm{Re}\,\alpha_i = 0$ for all i. Applying the same argument to $\sum_i \sqrt{-1}\,\alpha_i x_i = 0$, we obtain that $\mathrm{Im}\,\alpha_i = 0$ for all i. Then **the natural map $V^\iota \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow V$ is injective.**

Step 2: This map is also surjective. Indeed, for any $v \in V$, one has $\frac{1}{2}(v + \iota(v)) \in V^\iota$ and $\frac{\sqrt{-1}}{2}(v - \iota(v)) \in V^\iota$, hence v can b expressed as a linear combination of vectors from V^ι with complex coefficients. ■

Real structures on complex manifolds

DEFINITION: A map $\Psi : M \longrightarrow M$ on an almost complex manifold (M, I) is called **antiholomorphic** if $d\Psi(I) = -I$. A function f is called **antiholomorphic** if $\overline{f}$ is holomorphic.

EXERCISE: Prove that **an antiholomorphic function on M defines an antiholomorphic map from M to $\mathbb{C}$.**

EXERCISE: Let ι be a smooth map from a complex manifold M to itself. Prove that ι **is antiholomorphic if and only if $\iota^*(f)$ is antiholomorphic for any holomorphic function f on $U \subset M$.**

DEFINITION: A real structure on a complex manifold M is an antiholomorphic involution $\tau : M \longrightarrow M$.

EXAMPLE: Complex conjugation defines a real structure on $\mathbb{C}^n$.