Complex geometry

lecture 5: Real analytic manifolds

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Real analytic manifolds

DEFINITION: A real analytic function on an open set $U \subset \mathbb{R}^n$ is a function which admits a Taylor expansion near each point $x \in U$:

$$f(z_1 + t_1, z_2 + t_2, ..., z_n + t_n) = \sum_{i_1,...,i_n} a_{i_1,...,i_n} t_1^{i_1} t_2^{i_2} ... t_n^{i_n}.$$

(here we assume that the real numbers t_i satisfy $|t_i| < \varepsilon$, where ε depends on f and M).

REMARK: Clearly, **real analytic functions constitute a sheaf.**

DEFINITION: A **real analytic manifold** is a ringed space which is locally isomorphic to an open ball $B \subset \mathbb{R}^n$ with the sheaf of of real analytic functions.

Real structures on complex vector spaces

DEFINITION: An involution is a map $\iota: M \longrightarrow M$ such that $\iota^2 = \mathrm{Id}_M$. **A real structure** on a complex vector space $V = \mathbb{C}^n$ is an $\mathbb{R}$-linear involution $\iota: V \longrightarrow V$ such that $\iota(\lambda x) = \overline{\lambda}\iota(x)$ for any $\lambda \in \mathbb{C}$.

CLAIM: Let ι be a real structure on a complex vector space V, and $V^\iota \subset V$ the space of V-invariant vectors. **Then $\dim_{\mathbb{R}} V^\iota = \dim_{\mathbb{C}} V$, and the natural map $V^\iota \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow V$ is an isomorphism.**

Proof. Step 1: Let $x_1, ..., x_n$ be a basis in V^ι, and $\sum_i \alpha_i x_i = 0$ a linear relation in V, with $\alpha_i \in \mathbb{C}$. Then $0 = \iota\left(\sum_i \alpha_i x_i\right) = \sum_i \overline{\alpha}_i x_i$. Summing up these two relations, we obtain $\sum_i \operatorname{Re} \alpha_i x_i = 0$. Since x_i are linearly independent over $\mathbb{R}$, this implies $\operatorname{Re} \alpha_i = 0$ for all i. Applying the same argument to $\sum_i \sqrt{-1}\, \alpha_i x_i = 0$, we obtain that $\operatorname{Im} \alpha_i = 0$ for all i. Then **the natural map $V^\iota \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow V$ is injective.**

Step 2: This map is also surjective. Indeed, for any $v \in V$, one has $(v + \iota(v)) \in V^\iota$ and $(v - \iota(v)) \in V^\iota$, hence v can b expressed as a linear combination of vectors from V^ι with complex coefficients. ∎

Real structures on complex manifolds

DEFINITION: A smooth map $\Psi: M \longrightarrow N$ on an almost complex manifold (M, I) is called **antiholomorphic** if $d\Psi(I) = -I$. A function f is called **antiholomorphic** if $\overline{f}$ is holomorphic.

EXERCISE: Prove that **an antiholomorphic function on M defines an antiholomorphic map from M to $\mathbb{C}$.**

EXERCISE: Prove that a map $\Psi: M \longrightarrow N$ of almost complex manifolds is antiholomorphic **if and only if $\Psi^*(\Lambda^{0,1}(N)) \subset \Lambda^{1,0}(M)$.**

EXERCISE: Let ι be a smooth map from a complex manifold M to itself. Prove that ι **is antiholomorphic if and only if $\iota^*(f)$ is antiholomorphic for any holomorphic function f on $U \subset M$.**

DEFINITION: A real structure on a complex manifold M is an antiholomorphic involution $\tau: M \longrightarrow M$.

EXAMPLE: Complex conjugation defines a real structure on $\mathbb{C}^n$.

Fixed points of real structures on manifolds

PROPOSITION: Let M be a complex manifold and $\iota: M \longrightarrow M$ a real structure. Denote by M^ι the fixed point set of ι. Then, **for each $x \in M^\iota$ there exists a ι-invariant coordinate neighbourhood with holomorphic coordinates $z_1, ..., z_n$, such that $\iota^*(z_i) = \overline{z}_i$.**

Proof. Step 1: For each basis of 1-forms $\nu_1, ..., \nu_n \in \Lambda^{1,0}_x(M)$, there exists a set of holomorphic coordinate functions $u_1, ..., u_n$ such that $du_i|_x = \nu_i$. To obtain such a coordinate system, **we chose any coordinate system $v_1, ..., v_n$ and apply a linear transform mapping $dv_i|_x$ to ν_i.**

Step 2: The differential $d\iota$ acts on T_xM as a real structure. Using the structure theorem about real structures, we obtain that any real basis $\zeta_1, .., \zeta_n$ of $T^*_x M^\iota$ is a complex basis in the complex vector space $T^*_x M$. Then $\nu_i := \zeta_i + \sqrt{-1}\, I(\zeta_i)\}$ is a basis in $\Lambda^{1,0}_x(M)$. Choose the coordinate system $u_1, ..., u_n$ such that $du_i|_x = \nu_i$ (Step 1). **Replacing u_i by $z_i := u_i + \iota^*(\overline{u}_i)$, we obtain a holomorphic coordinate system z_i on M** (compare with Theorem 1 in Lecture 4) **which satisfies $\iota^*(z_i) = \overline{z}_i$.** ■

DEFINITION: Let $\{U_i\}$ be an complex atlas on M. Assume that any U_i intersecting M^ι satisfies the conclusion of this proposition. Then $\{U_i\}$ is called **compatible with the real structure**.

Real analytic manifolds and real structures

PROPOSITION: Let $M^\iota \subset M$ be a fixed point set of an antiholomorphic involution ι on a complex manifold M, $\{U_i\}$ a complex analytic atlas, and $\Psi_{ij}: U_{ij} \longrightarrow U_{ij}$ the gluing functions. Assume that the atlas U_i is compatible with the real structure, in the sense of the previous proposition. **Then all Ψ_{ij} are real analytic on M^ι, and define a real analytic atlas on the manifold M^ι.**

Proof: All gluing functions from one coordinate system compatible with the real structure to another **commute with ι, acting on coordinate functions as the complex conjugation.** This gives $\Psi_{ij}(\bar{z}_i) = \overline{\Psi_{ij}(z_i)}$. Therefore, Ψ_{ij} preserve M^ι, and are expressed by real-valued functions on M^ι. ■

Real analytic manifolds and real structures (2)

PROPOSITION: Any real analytic manifold can be obtained from this construction.

Proof. Step 1: Let $\{U_i\}$ be a locally finite atlas of a real analytic manifold M, and $\Psi_{ij}: U_{ij} \longrightarrow U_{ij}$ the gluing maps. We realize U_i as an open ball with compact closure in $\operatorname{Re}(\mathbb{C}^n) = \mathbb{R}^n$. By local finiteness, there are only finitely many such Ψ_{ij} for any given U_i. Denote by B_ε an open ball of radius ε in the n-dimensional real space $\operatorname{im}(\mathbb{C}^n)$.

Step 2: Let $\varepsilon > 0$ be a sufficiently small real number such that all Ψ_{ij} can be extended to gluing functions $\tilde{\Psi}_{ij}$ on the open sets $\tilde{U}_i := U_i \times B_\varepsilon \subset \mathbb{C}^n$. **Then $(\tilde{U}_i, \Psi_{ij})$ is an atlas for a complex manifold $M_{\mathbb{C}}$.** Since all Ψ_{ij} are real, they are preserved by the natural involution acting on B_ε as -1 and on U_i as identity. This involution defines a real structure on $M_{\mathbb{C}}$. Clearly, M is the set of its fixed points. ■

Complexification

DEFINITION: Let $M_{\mathbb{R}}$ be a real analytic manifold, and $M_{\mathbb{C}}$ a complex analytic manifold equipped with an antiholomorphic involution, such that $M_{\mathbb{R}}$ is the set of its fixed points. Then $M_{\mathbb{C}}$ is called **complexification** of $M_{\mathbb{R}}$.

DEFINITION: A tensor on a real analytic manifold is called **real analytic** if it is expressed locally by a sum of coordinate monomials with real analytic coefficients.

CLAIM: Let $M_{\mathbb{R}}$ be a real analytic manifold, $(M_{\mathbb{C}}, \iota)$ its complexification, and Φ a tensor on $M_{\mathbb{R}}$. **Then Φ is real analytic if and only if Φ can be extended to a holomorpic tensor $\Phi_{\mathbb{C}}$ in some neighbourhood of $M_{\mathbb{R}}$ inside $M_{\mathbb{C}}$.** Moreover, **Φ is real on $M_{\mathbb{R}}$ if $\iota^* \Phi_{\mathbb{C}} = \overline{\Phi}_{\mathbb{C}}$.**

Proof: The "if" part is clear, because every complex analytic tensor on $M_{\mathbb{C}}$ is by definition real analytic on $M_{\mathbb{R}}$.

Conversely, suppose that Φ is expressed in coordinates by a sum of tensorial monomials with real analytic coefficients f_i. Let $\{U_i\}$ be a cover of M, and $\tilde{U}_i := U_i \times B_\varepsilon$ the corresponding cover of a neighbourhood of $M_{\mathbb{R}}$ in $M_{\mathbb{C}}$ constructed above. Chosing ε sufficiently small, we can assume that the Taylor series giving coefficients of Φ converges on each $\tilde{U}_i$. **We define $\Phi_{\mathbb{C}}$ as the sum of these series.** ■

Categories

DEFINITION: A category $\mathcal{C}$ is a collection of data called "objects" and "morphisms between objects" which satisfies the axioms below.

DATA.

Objects: A class $\mathcal{O}\lfloor(\mathcal{C})$ of **objects** of $\mathcal{C}$.

Morphisms: For each $X, Y \in \mathcal{O}\lfloor(\mathcal{C})$, one has a set $\mathcal{M}\wr\nabla(X,Y)$ of **morphisms from X to Y**.

Composition of morphisms: For each $\varphi \in \mathcal{M}\wr\nabla(X,Y), \psi \in \mathcal{M}\wr\nabla(Y,Z)$ there exists **the composition** $\varphi \circ \psi \in \mathcal{M}\wr\nabla(X,Z)$

Identity morphism: For each $A \in \mathcal{O}\lfloor(\mathcal{C})$ there exists a morphism $\mathrm{Id}_A \in \mathcal{M}\wr\nabla(A,A)$.

AXIOMS.

Associativity of composition: $\varphi_1 \circ (\varphi_2 \circ \varphi_3) = (\varphi_1 \circ \varphi_2) \circ \varphi_3$.

Properties of identity morphism: For each $\varphi \in \mathcal{M}\wr\nabla(X,Y)$, one has $\mathrm{Id}_x \circ \varphi = \varphi = \varphi \circ \mathrm{Id}_Y$

Categories (2)

DEFINITION: Let $X, Y \in \mathcal{O}\lfloor(\mathcal{C})$ – objects of $\mathcal{C}$. A morphism $\varphi \in \mathcal{M}\wr\nabla(X, Y)$ is called **an isomorphism** if there exists $\psi \in \mathcal{M}\wr\nabla(Y, X)$ such that $\varphi \circ \psi = \mathrm{Id}_X$ and $\psi \circ \varphi = \mathrm{Id}_Y$. In this case, the objects X and Y are called **isomorphic**.

Examples of categories:
Category of sets: its morphisms are arbitrary maps.
Category of vector spaces: its morphisms are linear maps.
Categories of rings, groups, fields: morphisms are homomorphisms.
Category of topological spaces: morphisms are continuous maps.
Category of smooth manifolds: morphisms are smooth maps.

Functors

DEFINITION: Let $\mathcal{C}_1, \mathcal{C}_2$ be two categories. A **covariant functor** from $\mathcal{C}_1$ to $\mathcal{C}_2$ is the following set of data.

1. **A map** $F: \mathcal{O}\lfloor(\mathcal{C}_1) \longrightarrow \mathcal{O}\lfloor(\mathcal{C}_2)$.

2. **A map** $F: \mathcal{M}\wr\nabla(X,Y) \longrightarrow \mathcal{M}\wr\nabla(F(X),F(Y))$ **defined for any pair of objects** $X,Y \in \mathcal{O}\lfloor(\mathcal{C}_1)$**.**

These data define a functor if they are **compatible with compositions,** that is, satisfy $F(\varphi)\circ F(\psi) = F(\varphi\circ\psi)$ for any $\varphi \in \mathcal{M}\wr\nabla(X,Y)$ and $\psi \in \mathcal{M}\wr\nabla(Y,Z)$, and **map identity morphism to identity** morphism.

Small categories

REMARK: This way, one could speak of **category of all categories**, with categories as objects and functors as morphisms.

A caution To avoid set-theoretic complications, Grothendieck added another axiom to set theory, "universum axiom", postulating existence of "universum", a very big set, and worked with "small categories" – categories where the set of all objects and sets of morphisms belong to the universum. In this sense, "category of all categories" is not a "small category", because the set of its object (being comparable to the set of all subsets of the universum) is too big to fit in the universum.

In practice, mathematicians say "category" when they mean "small category", tacitly assuming that any given category is "small". This is why not many people call "category of all categories" a category: nobody wants to deal with set-theoretic complications.

Example of functors

A "natural operation" on mathematical objects is usually a functor. Examples:

1. A map $X \longrightarrow 2^X$ from the set X to the set of all subsets of X is a functor from the category $\mathcal{S}$ets of sets to itself.

2. A map $M \longrightarrow M^2$ mapping a topological space to its product with itself is a functor on topological spaces.

3. A map $V \longrightarrow V \oplus V$ is a functor on vector spaces; same for a map $V \longrightarrow V \otimes V$ or $V \longrightarrow (V \oplus V) \otimes V$.

4. **Identity functor** from any category to itself.

5. A map from topological spaces to $\mathcal{S}$ets, putting a topological space to the set of its connected components.

EXERCISE: Prove that it is a functor.

Equivalence of functors

DEFINITION: Let $X, Y \in \mathcal{O}\lfloor(\mathcal{C})$ be objects of a category $\mathcal{C}$. A mprphism $\varphi \in \mathcal{M}\wr\nabla(X,Y)$ is called **an isomorphism** if there exists $\psi \in \mathcal{M}\wr\nabla(Y,X)$ such that $\varphi \circ \psi = \mathrm{Id}_X$ and $\psi \circ \varphi = \mathrm{Id}_Y$. In this case X and Y are called **isomorphic**.

DEFINITION: Two functors $F, G : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ are called **equivalent** if for any $X \in \mathcal{O}\lfloor(\mathcal{C}_1)$ we are given an isomorphism $\Psi_X : F(X) \longrightarrow G(X)$, in such a way that for any $\varphi \in \mathcal{M}\wr\nabla(X,Y)$, one has $F(\varphi) \circ \Psi_Y = \Psi_X \circ G(\varphi)$.

REMARK: Such commutation relations are usually expressed by **commutative diagrams**. For example, the condition $F(\varphi) \circ \Psi_Y = \Psi_X \circ G(\varphi)$ is expressed by a commutative diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(\varphi)"] \arrow[d, "\Psi_X"'] & F(Y) \arrow[d, "\Psi_Y"] \\ G(X) \arrow[r, "G(\varphi)"] & G(Y) \end{tikzcd}$$

Equivalence of categories

DEFINITION: A functor $F: \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is called **equivalence of categories** if there exists a functor $G: \mathcal{C}_2 \longrightarrow \mathcal{C}_1$ such that the compositions $G \circ F$ and $G \circ F$ are equivaleent to the identity functors $\mathrm{Id}_{\mathcal{C}_1}$, $\mathrm{Id}_{\mathcal{C}_2}$.

REMARK: It is possible to show that this is equivalent to the following conditions: **F defines a bijection on the set of isomorphism classes of objects of $\mathcal{C}_1$ and $\mathcal{C}_2$, and a bijection**

$$\mathcal{M}\wr\nabla(X,Y) \longrightarrow \mathcal{M}\wr\nabla(F(X),F(Y)).$$

for each $X, Y \in \mathcal{O}\lfloor(\mathcal{C}_1)$.

REMARK: From the point of view of category theory, **equivalent categories are two instances of the same category** (even if the cardinality of corresponding sets of objects is different).

Germ of a complex manifold

DEFINITION: Let $K \subset M$ be a closed subset of a complex manifold, homeomorphic to $K_1 \subset M_1$, where M_1 is also a complex manifold. Fixing the homeomorphism $K \cong K_1$, we may identify these sets and consider K as a subset M_1. We say that M and M_1 **have the same germ in K** if there exist biholomorphic open subsets $U_1 \subset M_1$ and $U \subset M$ containing K, with the biholomorphism $\varphi: U \longrightarrow U_1$ identity on K.

DEFINITION: Germ of a manifold M in $K \subset M$ is an equivalence class of open subsets $U \subset M$ containing K, with this equivalence relation.

DEFINITION: Consider category $\mathcal{C}_\iota$, with objects complex manifolds (M, ι) equipped with a real structure, and morphisms holomorphic maps commuting with ι.

THEOREM: (Grauert) Category of real analytic manifolds is equivalent to the category of germs of $M \in \mathcal{C}_\iota$ in $M^\iota \subset M$.

EXERCISE: Prove this theorem.

Hans Grauert

Hans Grauert in Bonn, 2000

(8.02.1930 - 4.09.2011)

Extension of tensors to a complexification

Lemma 1: Let X be an open ball in $\mathbb{C}^n$ equipped with the standard anticomplex involution, $X_{\mathbb{R}} = X \cap \mathbb{R}^n$ its fixed point set, and α a holomorphic tensor on X vanishing in $X_{\mathbb{R}}$. **Then $\alpha = 0$.**

Proof: Any holomorphic function which vanishes on $\mathbb{R}^n$ has all its derivatives vanishing. Therefore its Taylor series vanish. Such a function vanishes on $\mathbb{C}^n$ by analytic continuation principle. This argument can be applied to all coefficients of α. ■

DEFINITION: An almost complex structure I on a real analytic manifold is **real analytic** if I is a real analytic tensor.

COROLLARY: Let (M, I) be a real analytic almost complex manifold, $M_{\mathbb{C}}$ its complexification, and $I_{\mathbb{C}} : TM_{\mathbb{C}} \longrightarrow TM_{\mathbb{C}}$ the holomorphic extension of I to $M_{\mathbb{C}}$. **Then $I_{\mathbb{C}}^2 = -\operatorname{Id}$.**

Proof: The tensor $I_{\mathbb{C}}^2 + \operatorname{Id}$ is holomorphic and vanishes on $M_{\mathbb{R}}$, hence the previous lemma can be applied. ■

Underlying real analytic manifold

REMARK: A complex analytic map $\Phi: \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is real analytic as a map $\mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$. Indeed, the coefficients of Φ are real and imaginary parts of holomorphic functions, and real and imaginary parts of holomorphic functions can be expressed as Taylor series of the real variables.

DEFINITION: Let M be a complex manifold. The **underlying real analytic manifold** $M_{\mathbb{R}}$ is the same manifold, with the same gluing functions, considered as real analytic maps.

REMARK: The sheaf of real analytic functions on $M_{\mathbb{R}}$ can be defined as **the sheaf of converging power series generated by holomorphic and antiholomorphic functions.** Indeed, such functions are real analytic in any of the real analytic map; conversely, **any real analytic function on $M_{\mathbb{R}}$ is a converging power serie on $\operatorname{Re} z_i, \operatorname{Im} z_i$, where z_i are holomorphic coordinates on M.**

Complexification of the underlying real analytic manifold

DEFINITION: Let M be a complex manifold. The **complex conjugate manifold** is the same manifold with almost complex structure $-I$ and anti-holomorphic functions on M holomorphic on $\overline{M}$.

CLAIM: Let M be an integrable almost complex manifold. Denote by $M_{\mathbb{R}}$ its underlying real analytic manifold. **Then a complexification of $M_{\mathbb{R}}$ can be given as $M_{\mathbb{C}} := M \times \overline{M}$,** with the anticomplex involution $\tau(x, y) = (y, x)$.

Proof: Clearly, the fixed point set of τ is the diagonal, identified with $M_{\mathbb{R}} = M$ as usual. Both holomorphic and antiholomorphic functions on $M_{\mathbb{R}}$ are obtained as restrictions of holomorphic functions from $M_{\mathbb{C}}$, hence the sheaf of real analytic functions on $M_{\mathbb{R}}$ is a subsheaf of $\mathcal{O}_{M_{\mathbb{C}}}$ of holomorphic functions on $M_{\mathbb{C}}$ restricted to $M_{\mathbb{R}}$. ∎