Complex geometry

lecture 8: Levi-Civita connection

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Connections

DEFINITION: Recall that **a connection** on a bundle B is an operator $\nabla : B \longrightarrow B \otimes \Lambda^1 M$ satisfying $\nabla(fb) = b \otimes df + f\nabla(b)$, where $f \longrightarrow df$ is de Rham differential. When X is a vector field, we denote by $\nabla_X(b) \in B$ the term $\langle \nabla(b), X\rangle$.

REMARK: A connection ∇ on B gives a connection $B^* \xrightarrow{\nabla^*} \Lambda^1 M \otimes B^*$ on the dual bundle, by the formula

$$d(\langle b, \beta\rangle) = \langle \nabla b, \beta\rangle + \langle b, \nabla^* \beta\rangle$$

These connections are usually denoted **by the same letter ∇.**

REMARK: For any tensor bundle $\mathcal{B}_1 := B^* \otimes B^* \otimes ... \otimes B^* \otimes B \otimes B \otimes ... \otimes B$ **a connection on B defines a connection on $\mathcal{B}_1$** using the Leibniz formula:

$$\nabla(b_1 \otimes b_2) = \nabla(b_1) \otimes b_2 + b_1 \otimes \nabla(b_2).$$

Parallel transport along the connection

THEOREM: Let B be a vector bundle with connection over $\mathbb{R}$. Then for each $x \in \mathbb{R}$ and each vector $b_x \in B|_x$ **there exists a unique section $b \in B$ such that $\nabla b = 0$, $b|_x = b_x$.**

Proof: This is existence and uniqueness of solutions of an ODE $\frac{db}{dt} + A(b) = 0$. ■

DEFINITION: Let $\gamma: [0,1] \longrightarrow M$ be a smooth path in M connecting x and y, and (B, ∇) a vector bundle with connection. Restricting (B, ∇) to $\gamma([0,1])$, we obtain a bundle with connection on an interval. Solve an equation $\nabla(b) = 0$ for $b \in B\big|_{\gamma([0,1])}$ and initial condition $b|_x = b_x$. This process is called **parallel transport** along the path via the connection. The vector $b_y := b|_y$ is called **vector obtained by parallel transport of b_x along γ**. **Holonomy group** of γ is the group of endomorphisms of the fiber B_x obtained from parallel transports along all paths starting and ending in $x \in M$

Parallel tensors

DEFINITION: Let B be a vector bundle, and $\Psi \in B^{\otimes i} \otimes (B^*)^{\otimes j}$ a tensor on B. We say that **connection ∇ preserves Ψ** if $\nabla(\Psi) = 0$. In this case we also say that Ψ is **parallel** with respect to the connection.

REMARK: $\nabla(\Psi) = 0$ is equivalent to Ψ being a solution of $\nabla(\Psi) = 0$ on each path γ. This means that **parallel transport preserves Ψ.**

We obtained

COROLLARY: A tensor is parallel if and only if it is holonomy invariant.

EXAMPLE: Orthogonal connection: given a positive definite form $h \in \operatorname{Sym}^2 B^*$ on B, a connection ∇ such that $\nabla(h) = 0$ is called **orthogonal**.

EXAMPLE: Suppose that (B, I) is a complex vector bundle equipped with a Hermitian metric h. A connection ∇ such that $\nabla(I) = \nabla(h) = 0$ is called **unitary**, or **Hermitian**.

Torsors and affine spaces

DEFINITION: A torsor over a group G is a space X equipped with a free and transitive action of G, $g, x \longrightarrow \rho(g, x)$.

DEFINITION: Morphism of torsors $(X, G, \rho) \xrightarrow{\Psi} (X', G', \rho')$ is a pair $\Psi_X : X \longrightarrow X', \Psi_G : \quad G \longrightarrow G'$, where Ψ_G is a group homomorphism satisfying $\Psi_X(\rho(g, x)) = \rho'(\Psi_G(g), \Psi_X(x))$ (that is, compatible with the map Ψ_X).

REMARK: This defines the category of torsors.

DEFINITION: Affine space is a torsor over a vector space V, which is called **linearization**. The action of V on A is denoted $a, v \longrightarrow a + v$.

EXAMPLE: Given two connections ∇ and ∇_1 on B, the difference $\nabla - \nabla_1$ is an $\operatorname{End}(B)$-valued 1-form. Converse is also true: for any $\operatorname{End}(B)$-valued 1-form $A \in \Lambda^1 M \otimes \operatorname{End}(B)$, the operator $\nabla + A$ is a connection. In other words, **the space of connections is an affine space over $\Lambda^1 M \otimes \operatorname{End}(B)$**.

Affine space of orthogonal connections

CLAIM: Let B be a bundle with a scalar product. Then **the space of orthogonal connections on B an affine space over $\Lambda^1 M \otimes \mathfrak{so}(B)$.**

Proof: Let $s \in B^* \otimes B^*$ be a 2-form on B. The action of $A := \nabla - \nabla_1$ on $B^* \otimes B^*$ is given by $A(s)(x,y) = -s(A(x),y) - s(x,A(y))$. Therefore, a difference A of orthogonal connections satisfies $h(A(x),y) = -h(x,A(y))$ for all $x, y \in B$. This is the same as $A \in \Lambda^1 M \otimes \mathfrak{so}(B)$. ■

Similarly one proves

CLAIM: Let B be a bundle with a Hermitian structure and a tensor Φ, and $\mathfrak{g} \subset \operatorname{End}(B)$ the Lie algebra of endomorphisms preserving Φ. Then **the space of connections on B preserving Φ is an affine space over $\Lambda^1 M \otimes \mathfrak{g}$.**

REMINDER: de Rham algebra

DEFINITION: Let $\Lambda^* M$ denote the vector bundle with the fiber $\Lambda^* T_x^* M$ at $x \in M$ ($\Lambda^* T^* M$ is the Grassman algebra of the cotangent space $T_x^* M$). The sections of $\Lambda^i M$ are called **differential i-forms**. The algebraic operation "wedge product" defined on differential forms is $C^\infty M$-linear; the space $\Lambda^* M$ of all differential forms is called **the de Rham algebra**.

REMARK: $\Lambda^0 M = C^\infty M$.

THEOREM: **There exists a unique operator $C^\infty M \xrightarrow{d} \Lambda^1 M \xrightarrow{d} \Lambda^2 M \xrightarrow{d} \Lambda^3 M \xrightarrow{d} \ldots$ satisfying the following properties**

1. On functions, d is equal to the differential.
2. $d^2 = 0$
3. $d(\eta \wedge \xi) = d(\eta) \wedge \xi + (-1)^{\tilde\eta} \eta \wedge d(\xi)$, where $\tilde\eta = 0$ where $\eta \in \lambda^{2i} M$ is **an even form,** and $\eta \in \lambda^{2i+1} M$ is **odd.**

DEFINITION: The operator d is called **de Rham differential**.

Cartan formula

CLAIM: For any $\eta \in \Lambda^1 M$, and $X, Y \in TM$ one has

$$d\eta(X,Y) = \eta([X,Y]) - \operatorname{Lie}_X(\eta(Y)) + \operatorname{Lie}_Y(\eta(X)).$$

Proof: Two sides of this equation define two operators $d, d_1 \Lambda^1 M \longrightarrow \Lambda^2 M$. Both operators satisfy the Leibniz rule $d(f\eta) = df \wedge d\eta + f d\eta$. When $\eta = df$ is exact, one has

$$\eta([X,Y]) - \operatorname{Lie}_X(\eta(Y)) + \operatorname{Lie}_Y(\eta(X)) = \\ = \operatorname{Lie}_{[X,Y]}(f) - \operatorname{Lie}_X \operatorname{Lie}_Y(f) + \operatorname{Lie}_Y \operatorname{Lie}_X(f) = 0$$

hence $d_1(\alpha) = 0$ on all closed forms. A map $\delta: \Lambda^1(M) \longrightarrow \Lambda^2(M)$ which vanishes on closed forms and satisfies the Leibniz rule is de Rham differential, which can be seen from the axiomatic definition of d. ■

Torsion

DEFINITION: Let ∇ be a connection on $\Lambda^1 M$,

$$\Lambda^1 \xrightarrow{\nabla} \Lambda^1 M \otimes \Lambda^1 M.$$

Torsion of ∇ $T_\nabla: \Lambda^1 M \longrightarrow \Lambda^2 M$ is a map $\nabla \circ \mathsf{Alt} - d$, where $\mathsf{Alt}: \Lambda^1 M \otimes \Lambda^1 M \longrightarrow \Lambda^2 M$ is exterior multiplication.

REMARK:

$$\begin{aligned} T_\nabla(f\eta) &= \mathsf{Alt}(f\nabla\eta + df \otimes \eta) - d(f\eta) \\ &= f\left[\mathsf{Alt}(\nabla\eta) - d\eta\right] + df \wedge \eta - df \wedge \eta = fT_\nabla(\eta). \end{aligned}$$

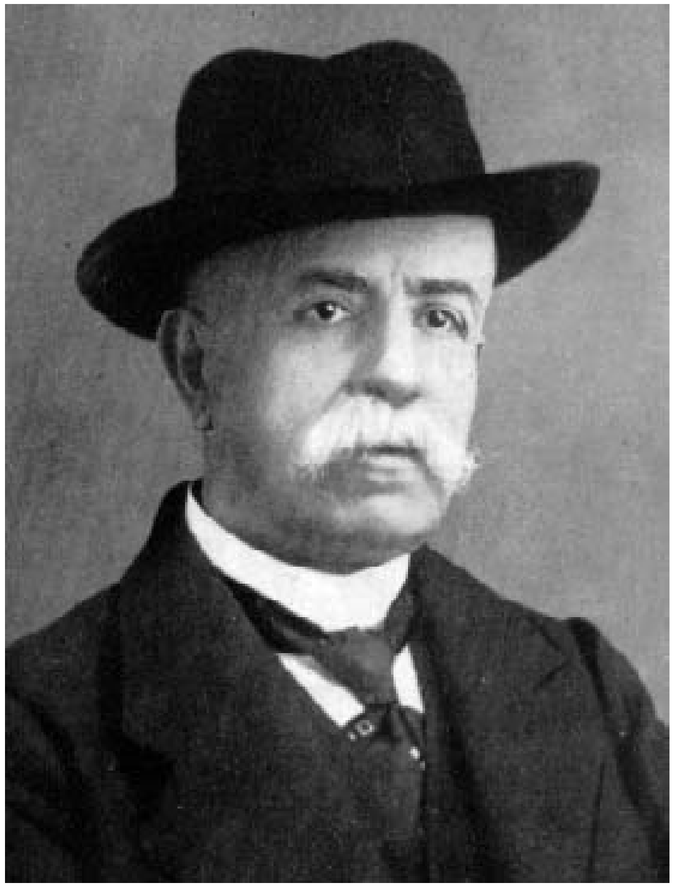
Therefore T_∇ is linear.

DEFINITION: Let (M,g) be a Riemannian manifold. A connection ∇ on TM is called **orthogonal** if $\nabla(g) = 0$, and **Levi-Civita connection** if it is orthogonal and has zero torsion.

THEOREM: ("the fundamental theorem of Riemannian geometry") Every Riemannian manifold admits a Levi-Civita connection, and it is unique.

Will be proven later today.

Gregorio Ricci-Curbastro, Tullio Levi-Civita



Gregorio Ricci-Curbastro,
1853-1925

Tullio Levi-Civita,
1873-1941

...With his former student Tullio Levi-Civita, he wrote his most famous single publication, a pioneering work on the calculus of tensors, signing it as Gregorio Ricci. This appears to be the only time that Ricci-Curbastro used the shortened form of his name in a publication, and continues to cause confusion.

Torsion and commutator of vector fields

REMARK: Cartan formula gives

$$\begin{aligned}T_\nabla(\eta)(X,Y) =&\nabla_X(\eta)(Y) - \nabla_Y(\eta)(X) - d\eta(X,Y)\\ =&\nabla_X(\eta)(Y) - \nabla_Y(\eta)(X) - \eta([X,Y]) - \operatorname{Lie}_X(\eta(Y)) + \operatorname{Lie}_Y(\eta(X)).\end{aligned}$$

On the other hand, $\nabla_X(\eta)(Y) = \operatorname{Lie}_X(\eta(Y)) - \eta(\nabla_X(Y))$. Comparing the equations, we obtain

$$T_\nabla(\eta)(X,Y) = \eta\Big(\nabla_X(Y) - \nabla_Y(X) - [X,Y]\Big).$$

Torsion is often defined as a map $\Lambda^2 TM \longrightarrow TM$ using the formula $\nabla_X(Y) - \nabla_Y(X) - [X,Y]$.

We have just proved

CLAIM: The torsion tensor $\nabla_X(Y) - \nabla_Y(X) - [X,Y]$ is dual to the torsion $\nabla \circ \operatorname{Alt} - d: \Lambda^1 M \longrightarrow \Lambda^2 M$ defined above. ■

Linearization of the torsion

REMARK: Consider the space $\mathscr{A}(\Lambda^1 M)$ of connections on $\Lambda^1 M$. The torsion defines an affine map

$$\mathscr{A}(\Lambda^1 M) \longrightarrow \operatorname{Hom}(\Lambda^1 M, \Lambda^2 M) = TM \otimes \Lambda^2 M.$$

because $T(\nabla+\alpha) = T(\nabla) + \operatorname{Alt}_{12}(\alpha)$, where $\operatorname{Alt}_{12}: \Lambda^1 M \otimes \operatorname{End}(\Lambda^1 M) \longrightarrow \Lambda^2 M \otimes TM$ is antisymmetrization in the first two indices.

DEFINITION: **Liearized torsion** is a map

$$T_{lin}: \Lambda^1(M) \otimes \Lambda^1(M) \otimes TM \longrightarrow \Lambda^2 M \otimes TM$$

obtained as a linearization of the torsion map. **It is equal to** Alt_{12}**.**

Existence of orthogonal connections

CLAIM: Let B be a vector bundle equipped with a scalar product. **Then B admits an orthogonal connection**.

Proof: Chose a covering $\{U_i\}$, such that B is trivial on each U_i and admits an orthonormal basis in each U_i. On each U_i we chose a connection ∇_i preserving this basis. Let ψ_i be a partition of unit subjugated to $\{U_i\}$. Then **the formula $\nabla(b) := \sum \nabla_i(\psi_i b)$ defines an orthogonal connection**.

THEOREM: ("the fundamental theorem of Riemannian geometry") Every Riemannian manifold admits a Levi-Civita connection, and it is unique.

Proof: See the next slide.

Levi-Civita connection (existence and uniqueness)

Proof. Step 1: Chose an orthogonal connection ∇_0 on $\Lambda^1 M$. The space $\mathscr{A}$ of orthogonal connections is affine and **its linearization is $\Lambda^1 M \otimes \mathfrak{so}(TM)$.** We shall identify $\mathfrak{so}(TM)$ and $\Lambda^2 M$. Then $\mathscr{A}$ **is an affine space over $\Lambda^1 M \otimes \Lambda^2 M$.**

Step 2: Then the linearized torsion map is

$$T_{lin}: \ \Lambda^1 M \otimes \mathfrak{so}(TM) = \Lambda^1(M) \otimes \Lambda^2 M \xrightarrow{\mathsf{Alt}_{12}} \Lambda^2 M \otimes \Lambda^1 M = \Lambda^2 M \otimes TM.$$

It is an isomorphism. Indeed, on the right and on the left there are bundles of the same rank, hence it would suffice to show that $T_{lin} = \mathsf{Alt}_{12}$ is injective. However, if $\eta \in \ker T_{lin}$, it is a form which is symmetric on first two arguments and antisymmetric on the second two, giving $\eta(x,y,z) = \eta(y,x,z) = -\eta(y,z,x)$. This gives $\sigma(\eta) = -\eta$, where σ is a cyclic permutation of the arguments. Since $\sigma^3 = 1$, this implies $\eta = 0$.

Step 3: We have shown that **an orthogonal connection is uniquely determined by its torsion**. Indeed, torsion map is an isomorphism of affine spaces.

Step 4: Let $\nabla := \nabla_0 - T_{lin}^{-1}(T_{\nabla_0})$. Then $T_\nabla = T_{\nabla_0} - T_{lin}(T_{lin}^{-1}(T_{\nabla_0})) = 0$, hence **$\nabla$ is torsion-free.** ■