Complex geometry

lecture 9: Symplectic connections

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Torsion (reminder)

DEFINITION: Let ∇ be a connection on $\Lambda^1 M$,

$$\Lambda^1 \xrightarrow{\nabla} \Lambda^1 M \otimes \Lambda^1 M.$$

Torsion of ∇ $T_\nabla : \Lambda^1 M \longrightarrow \Lambda^2 M$ is a map $\nabla \circ \mathrm{Alt} - d$, where $\mathrm{Alt} : \Lambda^1 M \otimes \Lambda^1 M \longrightarrow \Lambda^2 M$ is exterior multiplication.

REMARK:

$$\begin{aligned} T_\nabla(f\eta) &= \mathrm{Alt}(f\nabla\eta + df \otimes \eta) - d(f\eta) \\ &= f\left[\mathrm{Alt}(\nabla\eta) - d\eta\right] + df \wedge \eta - df \wedge \eta = fT_\nabla(\eta). \end{aligned}$$

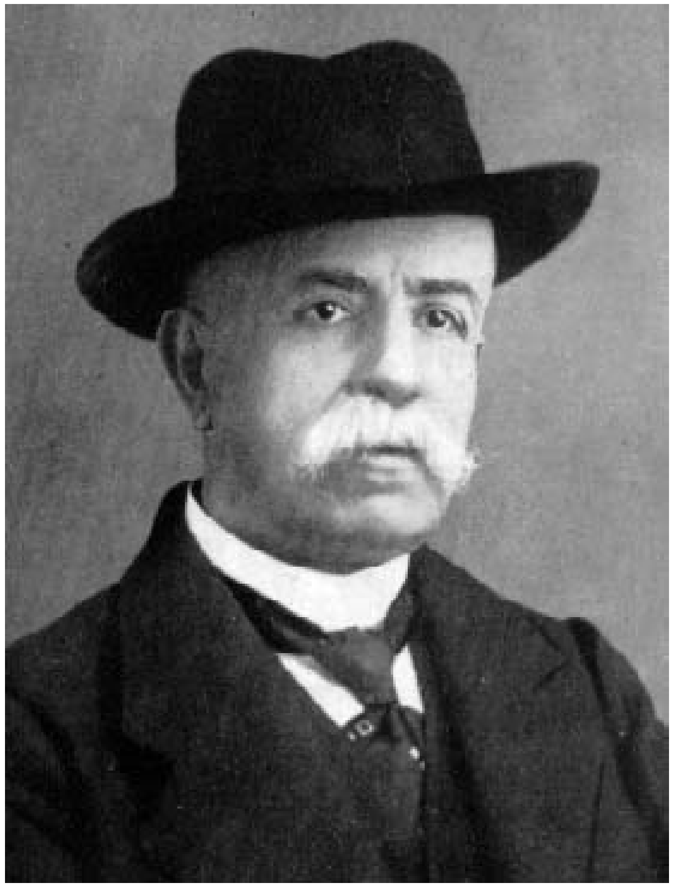
Therefore T_∇ is linear.

DEFINITION: Let (M, g) be a Riemannian manifold. A connection ∇ on TM is called **orthogonal** if $\nabla(g) = 0$, and **Levi-Civita connection** if it is orthogonal and has zero torsion.

THEOREM: ("the fundamental theorem of Riemannian geometry") Every Riemannian manifold admits a Levi-Civita connection, and it is unique.

Proof: Lecture 8.

Gregorio Ricci-Curbastro, Tullio Levi-Civita



Gregorio Ricci-Curbastro, 1853-1925

Tullio Levi-Civita, 1873-1941

...With his former student Tullio Levi-Civita, he wrote his most famous single publication, a pioneering work on the calculus of tensors, signing it as Gregorio Ricci. This appears to be the only time that Ricci-Curbastro used the shortened form of his name in a publication, and continues to cause confusion.

Torsion and commutator of vector fields (reminder)

REMARK: Cartan formula gives

$$\begin{aligned} T_\nabla(\eta)(X,Y) =& \nabla_X(\eta)(Y) - \nabla_Y(\eta)(X) - d\eta(X,Y) \\ =& \nabla_X(\eta)(Y) - \nabla_Y(\eta)(X) - \eta([X,Y]) - \operatorname{Lie}_X(\eta(Y)) + \operatorname{Lie}_Y(\eta(X)). \end{aligned}$$

On the other hand, $\nabla_X(\eta)(Y) = \operatorname{Lie}_X(\eta(Y)) - \eta(\nabla_X(Y))$. Comparing the equations, we obtain

$$T_\nabla(\eta)(X,Y) = \eta\Big(\nabla_X(Y) - \nabla_Y(X) - [X,Y]\Big).$$

Torsion is often defined as a map $\Lambda^2 TM \longrightarrow TM$ using the formula $\nabla_X(Y) - \nabla_Y(X) - [X,Y]$.

We have just proved

CLAIM: The torsion tensor $\nabla_X(Y) - \nabla_Y(X) - [X,Y]$ is dual to the torsion $\nabla \circ \operatorname{Alt} - d : \Lambda^1 M \longrightarrow \Lambda^2 M$ defined above. ■

Linearization of the torsion (reminder)

REMARK: Consider the space $\mathscr{A}(\Lambda^1 M)$ of connections on $\Lambda^1 M$. The torsion defines an affine map

$$\mathscr{A}(\Lambda^1 M) \longrightarrow \operatorname{Hom}(\Lambda^1 M, \Lambda^2 M) = TM \otimes \Lambda^2 M.$$

because $T(\nabla+\alpha) = T(\nabla) + \operatorname{Alt}_{12}(\alpha)$, where $\operatorname{Alt}_{12}: \Lambda^1 M \otimes \operatorname{End}(\Lambda^1 M) \longrightarrow \Lambda^2 M \otimes TM$ is antisymmetrization in the first two indices.

DEFINITION: **Linearized torsion** is a map

$$T_{\text{lin}}: \Lambda^1(M) \otimes \Lambda^1(M) \otimes TM \longrightarrow \Lambda^2 M \otimes TM$$

obtained as a linearization of the torsion map. **It is equal to Alt_{12}.**

Existence of orthogonal connections (reminder)

CLAIM: Let B be a vector bundle equipped with a scalar product. **Then B admits an orthogonal connection**.

Proof: Chose a covering $\{U_i\}$, such that B is trivial on each U_i and admits an orthonormal basis in each U_i. On each U_i we chose a connection ∇_i preserving this basis. Let ψ_i be a partition of unit subjugated to $\{U_i\}$. Then **the formula $\nabla(b) := \sum \nabla_i(\psi_i b)$ defines an orthogonal connection**.

EXERCISE: Let $\omega \in \Lambda^2 B^*$ be a non-degenerate skew-symmetric 2-form on B. Use the same argument to prove that there exists a connection $\nabla : B \longrightarrow B \otimes \Lambda^1 M$ such that $\nabla(\omega) = 0$.

THEOREM: ("the fundamental theorem of Riemannian geometry") Every Riemannian manifold admits a Levi-Civita connection, and it is unique.

Proof: See the next slide.

Levi-Civita connection (reminder)

Proof. Step 1: Chose an orthogonal connection ∇_0 on $\Lambda^1 M$. The space $\mathcal{A}$ of orthogonal connections is affine and **its linearization is $\Lambda^1 M \otimes \mathfrak{so}(TM)$.** We shall identify $\mathfrak{so}(TM)$ and $\Lambda^2 M$. Then $\mathcal{A}$ **is an affine space over $\Lambda^1 M \otimes \Lambda^2 M$**.

Step 2: Then the linearized torsion map is

$$T_{\mathsf{lin}}: \Lambda^1 M \otimes \mathfrak{so}(TM) = \Lambda^1(M) \otimes \Lambda^2 M \xrightarrow{\mathsf{Alt}_{12}} \Lambda^2 M \otimes \Lambda^1 M = \Lambda^2 M \otimes TM.$$

It is an isomorphism. Indeed, on the right and on the left there are bundles of the same rank, hence it would suffice to show that $T_{\mathsf{lin}} = \mathsf{Alt}_{12}$ is injective. However, if $\eta \in \ker T_{\mathsf{lin}}$, it is a form which is symmetric on first two arguments and antisymmetric on the second two, giving $\eta(x,y,z) = \eta(y,x,z) = -\eta(y,z,x)$. This gives $\sigma(\eta) = -\eta$, where σ is a cyclic permutation of the arguments. Since $\sigma^3 = 1$, this implies $\eta = 0$.

Step 3: We have shown that **an orthogonal connection is uniquely determined by its torsion**. Indeed, torsion map is an isomorphism of affine spaces.

Step 4: Let $\nabla := \nabla_0 - T_{\mathsf{lin}}^{-1}(T_{\nabla_0})$. Then $T_\nabla = T_{\nabla_0} - T_{\mathsf{lin}}(T_{\mathsf{lin}}^{-1}(T_{\nabla_0})) = 0$, hence **$\nabla$ is torsion-free.** ■

Lie algebra and tensors (reminder)

DEFINITION: Let V be a representation of a Lie algebra $\mathfrak{g}$. **Then V^* is also a representation;** the action of $\mathfrak{g}$ on V^* is given by the formula $\langle g(x), \lambda\rangle = -\langle x, g(\lambda)\rangle$, for all $x \in V, \lambda \in V^*$. A tensor product of two $\mathfrak{g}$-representations V_1, V_2 is also a $\mathfrak{g}$-representation, with the action of $\mathfrak{g}$ defined by $g(x \otimes y) = g(x) \otimes y + x \otimes g(y)$. This defines the action of $\mathfrak{g}$ on all tensor powers $V^{\otimes i} \otimes (V^*)^{\otimes j}$, which are called **the tensor representations** of $\mathfrak{g}$. We say that $\mathfrak{g}$ **preserves a tensor Φ** if $g(\Psi) = 0$ for all $g \in \mathfrak{g}$.

EXAMPLE: The algebra of all $g \in \operatorname{End}(V)$ preserving a non-degenerate bilinear symmetric form $h \in \operatorname{Sym}^2(V^*)$ is called **orthogonal algebra**, denoted $\mathfrak{so}(V,h)$ or $\mathfrak{so}(V)$. Since $g \in \mathfrak{so}(V)$ if and only if $h(g(x), y) = -h(x, g(y))$, **$\mathfrak{so}(V)$ is represented by antisymmetric matrices.**

CLAIM: Let $h \in \operatorname{Sym}^2(V^*)$ be a non-degenerate bilinear symmetric form. Using h, we identify V and V^*. This gives an isomorphism $V^* \otimes V^* \xrightarrow{\tau} V^* \otimes V = \operatorname{End}(V)$. **Then $\tau(\Lambda^2 V^*) = \mathfrak{so}(V)$.**

Proof: For any $f \in \operatorname{End}(V)$, the 2-form $\tau^{-1}(f)$ is written as $x, y \longrightarrow h(f(x), y)$. By definition, $f \in \mathfrak{so}(V)$ means that $h(f(x), y) = -h(x, f(y))$ and this happens if and only if $\tau^{-1}(f)$ is antisymmetric. ■

The Lie algebra $\mathfrak{u}(V)$

EXAMPLE: Let (V, I) be a real vector space with a complex structure map $I: V \longrightarrow V$, $I^2 = -\mathrm{Id}$, and a Hermitian (that is, I-invariant) scalar product. Define **the unitary Lie algebra** $\mathfrak{u}(V) = \{f \in \mathrm{End}(V) \mid f(I) = f(h) = 0\}$. This is the same as the space of I-invariant orthogonal matrices.

CLAIM: Consider the natural map $V^* \otimes V^* \xrightarrow{\tau} V^* \otimes V = \mathrm{End}(V)$ associated with h. **Then $\tau(\Lambda^{1,1}(V^*)) = \mathfrak{u}(V)$.**

Proof: The isomorphism τ is I-invariant, because h is I-invariant. **Then $\tau^{-1}(\mathfrak{u}(V))$ is the space of I-invariant 2-forms,** which is precisely $\Lambda^{1,1}(V^*)$.
■

COROLLARY: Let B be a bundle with a Hermitian structure product. Then **the space of orthogonal connections on B an affine space over $\Lambda^1 M \otimes \mathfrak{u}(B)$.**

The space of intrinsic torsion

REMARK: Let Φ be a tensor on a manifold, and ∇ a connection preserving Φ. Denote by $\mathfrak{a}(M) \subset \operatorname{End}(TM)$ the bundle of Lie algebras consisting of all $A \in \operatorname{End}(TM)$ such that $A(\Phi) = 0$. Clearly, a connection ∇_1 preserves Φ if and only if $\nabla - \nabla_1 \in \Lambda^1(M) \otimes \mathfrak{a}(M)$. In other words, **connections preserving Φ are an affine space over $\Lambda^1(M) \otimes \mathfrak{a}(M)$.**

DEFINITION: Consider the linearized torsion operator $\operatorname{Alt}_{12} : \Lambda^1(M) \otimes \mathfrak{a}(M) \longrightarrow \Lambda^2(M) \otimes TM$. The quotient bundle

$$\mathscr{T}_{\mathfrak{a}} := \frac{\Lambda^2(M) \otimes TM}{\operatorname{Alt}_{12}(\Lambda^1(M) \otimes \mathfrak{a}(M))}$$

is called **the space of intrinsic torsion for $\mathfrak{a}(M)$-valued connections**.

DEFINITION: Let Φ be a tensor on a manifold, and ∇ a connection preserving Φ. **Intrinsic torsion** of Φ is the image of the torsion of ∇ in $\mathscr{T}_{\mathfrak{a}}$.

Intrinsic torsion

THEOREM: Let Φ be a tensor on a manifold, ∇ a connection preserving Φ, and $\tau(\Phi)$ the intrinsic torsion. **Then $\tau(\Phi)$ is independent from the choice of ∇.** Moreover, M admits a torsion-free connection preserving Φ **if and only if $\tau(\Phi) = 0$.**

Proof. Step 1: For any ∇ and ∇' preserving Φ, and $A := \nabla - \nabla'$, one has $A \in \Lambda^1(M) \otimes \mathfrak{a}(M)$, hence $T_\nabla - T_{\nabla'} \in \mathrm{Alt}_{12}(\Lambda^1(M) \otimes \mathfrak{a}(M))$. Therefore, T_∇ represents the same vector in $\mathcal{T}_{\mathfrak{a}}$ as $T_{\nabla'}$

Step 2: The map $\nabla \mapsto T_\nabla$ takes an affine space of all connections preserving Φ and puts it to an affine subspace $W \subset \Lambda^2(M) \otimes TM$. The linearization of W is the image of T_{lin}, hence **W is an affine space $\mathrm{im}(T_{\mathrm{lin}}) + T_\nabla$.** It contains zero if and only if $T_\nabla \in \mathrm{im}(T_{\mathrm{lin}})$. ■

EXAMPLE: The space of intrinsic torsion for $\mathfrak{so}(TM)$ is zero **(prove it).**

EXAMPLE: The space of intrinsic torsion for the symplectic Lie algebra $\mathfrak{sp}(TM)$ is **naturally identified with the space $\Lambda^3(M)$** **(this is proven later today).**

Symplectic connections

DEFINITION: When $B = \Lambda^1 M$, consider the exterior multiplication map $\operatorname{Alt}: \Lambda^i M \otimes \Lambda^1 M \longrightarrow \Lambda^{i+1} M$. Define **the torsion map** $T_\nabla(\eta) := \operatorname{Alt}(\nabla(\eta)) - d\eta$. Then T_∇ is equal to torsion on $\Lambda^1 M$ and satisfies the Leibnitz identity:

$$T_\nabla(\lambda \wedge \mu) = T_\nabla(\lambda) \wedge \mu + (-1)^{\tilde{\lambda}} \lambda \wedge T_\nabla(\mu) \qquad (**)$$

DEFINITION: An almost symplectic structure on a manifold is a non-degenerate 2-form.

EXERCISE: Let (M, ω) be an almost symplectic manifold. Prove that there exists a connection ∇ on TM such that $\nabla(\omega) = 0$. We call such connection **a symplectic connection**.

Lemma 1: Let $\omega \in \Lambda^2 M$ be an almost symplectic structure, and ∇ a symplectic connection. Using ω, we will identify TM and $\Lambda^1 M$, and then we can consider the torsion tensor $\mathfrak{T} \in \Lambda^2 M \otimes TM$ of ∇ as $\tau \in \Lambda^2 M \otimes \Lambda^1 M$. Let $\rho := \operatorname{Alt}(\tau)$. **Then $d\omega = -2\rho$.**

Proof: Clearly, $T_\nabla(\omega) = -d\omega$, because $\nabla(\omega) = 0$ and $T_\nabla(\omega) = \operatorname{Alt}(\nabla(\omega)) - d\omega$. By (**), we have $T_\nabla(\omega) = \operatorname{Alt}(A_1(\omega \otimes \mathfrak{T}) - A_2(\omega \otimes \mathfrak{T}))$, where A_i is the convolution of i-th component of $\omega \otimes T_\nabla$ and the last, taking $\Lambda^2 M \otimes \Lambda^2 M \otimes TM$ to $\Lambda^2 M \otimes \Lambda^1 M$ and $\Lambda^1 M \otimes \Lambda^2 M$. Clearly, $\operatorname{Alt}(A_1(\omega \otimes T_\nabla)) = -\operatorname{Alt}(A_2(\omega \otimes T_\nabla)) = \rho$. This gives $T_\nabla(\omega) = d\omega = -2\rho$. ■

Torsion of almost symplectic structures

Theorem 1: Let (M,ω) be an almost symplectic manifold, and ∇ a symplectic connection. Denote its torsion by $T_\nabla \in \Lambda^2 M \otimes TM$. Using the form ω, we identify TM and $\Lambda^1 M$ and consider T_∇ as a section $\tau \in \Lambda^2 M \otimes \Lambda^1 M$. Denote by Alt_{123} the multiplication map $\Lambda^2 M \otimes \Lambda^1 M \longrightarrow \Lambda^3 M$. **Then** $\mathsf{Alt}_{123}(\tau) = -\frac{1}{2} d\omega$**.** Moreover, **any tensor** $\mathfrak{T} \in \Lambda^2 M \otimes \Lambda^1 M$ **such that** $\mathsf{Alt}_{123}(\mathfrak{T}) = -\frac{1}{2} d\omega$ **can be realized as a torsion of a symplectic connection.**

Proof. Step 1: Let $\mathfrak{sp}(TM)$ be the Lie algebra of all tensors $a \in \operatorname{End}(TM)$ such that $\omega(a(x),y) = -\omega(x,a(y))$. The same argument as the one used to show $\mathfrak{so}(TM) = \Lambda^2 M$ shows that $\mathfrak{sp}(TM) = \operatorname{Sym}^2(\Lambda^1 M)$.

Step 2: Under this identification, the linearized torsion map $T_{\mathsf{lin}} : \Lambda^1 M \otimes \mathfrak{sp}(TM) \longrightarrow \Lambda^2 M \otimes TM$ becomes $\mathsf{Alt}_{12} : \Lambda^1 M \otimes \operatorname{Sym}^2(\Lambda^1 M) \longrightarrow \Lambda^2 M \otimes \Lambda^1 M$. Kernel of this map is clearly $\operatorname{Sym}^3(\Lambda^1 M)$. This gives an exact sequence **(check it).**

$$0 \longrightarrow \operatorname{Sym}^3(\Lambda^1 M) \hookrightarrow \Lambda^1 M \otimes \operatorname{Sym}^2(\Lambda^1 M) \xrightarrow{\mathsf{Alt}_{12}} \Lambda^2 M \otimes \Lambda^1 M \xrightarrow{\mathsf{Alt}_{123}} \Lambda^3 M \longrightarrow 0.$$

We identified $\Lambda^3 M$ with the space of intrinsic torsion for $\mathfrak{sp}(TM)$.

Step 3: $\mathsf{Alt}_{123}(\tau) = -\frac{1}{2} d\omega$ (Lemma 1). This is precisely the intrinsic torsion of ∇. ■