Complex geometry

lecture 10: Bismut connection

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Lie algebra and tensors (reminder)

DEFINITION: Let V be a representation of a Lie algebra $\mathfrak{g}$. **Then V^* is also a representation;** the action of $\mathfrak{g}$ on V^* is given by the formula $\langle g(x), \lambda\rangle = -\langle x, g(\lambda)\rangle$, for all $x \in V, \lambda \in V^*$. A tensor product of two $\mathfrak{g}$-representations V_1, V_2 is also a $\mathfrak{g}$-representation, with the action of $\mathfrak{g}$ defined by $g(x \otimes y) = g(x) \otimes y + x \otimes g(y)$. This defines the action of $\mathfrak{g}$ on all tensor powers $V^{\otimes i} \otimes (V^*)^{\otimes j}$, which are called **the tensor representations** of $\mathfrak{g}$. We say that $\mathfrak{g}$ **preserves a tensor Φ** if $g(\Psi) = 0$ for all $g \in \mathfrak{g}$.

EXAMPLE: The algebra of all $g \in \operatorname{End}(V)$ preserving a non-degenerate bilinear symmetric form $h \in \operatorname{Sym}^2(V^*)$ is called **orthogonal algebra**, denoted $\mathfrak{so}(V, h)$ or $\mathfrak{so}(V)$. Since $g \in \mathfrak{so}(V)$ if and only if $h(g(x), y) = -h(x, g(y))$, **$\mathfrak{so}(V)$ is represented by antisymmetric matrices.**

CLAIM: Let $h \in \operatorname{Sym}^2(V^*)$ be a non-degenerate bilinear symmetric form. Using h, we identify V and V^*. This gives an isomorphism $V^* \otimes V^* \xrightarrow{\tau} V^* \otimes V = \operatorname{End}(V)$. **Then $\tau(\Lambda^2 V^*) = \mathfrak{so}(V)$.**

Proof: For any $f \in \operatorname{End}(V)$, the 2-form $\tau^{-1}(f)$ is written as $x, y \longrightarrow h(f(x), y)$. By definition, $f \in \mathfrak{so}(V)$ means that $h(f(x), y) = -h(x, f(y))$ and this happens if and only if $\tau^{-1}(f)$ is antisymmetric. ■

The Lie algebra $\mathfrak{u}(V)$ (reminder)

EXAMPLE: Let (V,I) be a real vector space with a complex structure map $I: V \longrightarrow V$, $I^2 = -\operatorname{Id}$, and a Hermitian (that is, I-invariant) scalar product. Define **the unitary Lie algebra** $\mathfrak{u}(V) = \{f \in \operatorname{End}(V) \ \mid \ f(I) = f(h) = 0\}$. This is the same as the space of I-invariant orthogonal matrices.

CLAIM: Consider the natural map $V^* \otimes V^* \xrightarrow{\tau} V^* \otimes V = \operatorname{End}(V)$ associated with h. **Then $\tau(\Lambda^{1,1}(V^*)) = \mathfrak{u}(V)$.**

Proof: The isomorphism τ is I-invariant, because h is I-invariant. **Then $\tau^{-1}(\mathfrak{u}(V))$ is the space of I-invariant 2-forms,** which is precisely $\Lambda^{1,1}(V^*)$.
■

COROLLARY: Let B be a bundle with a Hermitian structure product. Then **the space of orthogonal connections on B an affine space over $\Lambda^1 M \otimes \mathfrak{u}(B)$.**

The space of intrinsic torsion (reminder)

REMARK: Let Φ be a tensor on a manifold, and ∇ a connection preserving Φ. Denote by $\mathfrak{a}(M) \subset \operatorname{End}(TM)$ the bundle of Lie algebras consisting of all $A \in \operatorname{End}(TM)$ such that $A(\Phi) = 0$. Clearly, a connection ∇_1 preserves Φ if and only if $\nabla - \nabla_1 \in \Lambda^1(M) \otimes \mathfrak{a}(M)$. In other words, **connections preserving Φ are an affine space over $\Lambda^1(M) \otimes \mathfrak{a}(M)$.**

DEFINITION: Consider the linearized torsion operator $\operatorname{Alt}_{12} : \Lambda^1(M) \otimes \mathfrak{a}(M) \longrightarrow \Lambda^2(M) \otimes TM$. The quotient bundle

$$\mathcal{T}_{\mathfrak{a}} := \frac{\Lambda^2(M) \otimes TM}{\operatorname{Alt}_{12}(\Lambda^1(M) \otimes \mathfrak{a}(M))}$$

is called **the space of intrinsic torsion for $\mathfrak{a}(M)$-valued connections**.

DEFINITION: Let Φ be a tensor on a manifold, and ∇ a connection preserving Φ. **Intrinsic torsion** of Φ is the image of the torsion of ∇ in $\mathcal{T}_{\mathfrak{a}}$.

Intrinsic torsion (reminder)

THEOREM: Let Φ be a tensor on a manifold, ∇ a connection preserving Φ, and $\tau(\Phi)$ the intrinsic torsion. **Then $\tau(\Phi)$ is independent from the choice of ∇.** Moreover, M admits a torsion-free connection preserving Φ **if and only if $\tau(\Phi) = 0$.**

Proof. Step 1: For any ∇ and ∇' preserving Φ, and $A := \nabla - \nabla'$, one has $A \in \Lambda^1(M) \otimes \mathfrak{a}(M)$, hence $T_\nabla - T_{\nabla'} \in \operatorname{Alt}_{12}(\Lambda^1(M) \otimes \mathfrak{a}(M))$. Therefore, T_∇ represents the same vector in $\mathcal{T}_{\mathfrak{a}}$ as $T_{\nabla'}$

Step 2: The map $\nabla \mapsto T_\nabla$ takes an affine space of all connections preserving Φ and puts it to an affine subspace $W \subset \Lambda^2(M) \otimes TM$. The linearization of W is the image of T_{lin}, hence **W is an affine space $\operatorname{im}(T_{\text{lin}}) + T_\nabla$.** It contains zero if and only if $T_\nabla \in \operatorname{im}(T_{\text{lin}})$. ■

EXAMPLE: The space of intrinsic torsion for $\mathfrak{so}(TM)$ is zero **(prove it).**

EXAMPLE: The space of intrinsic torsion for the symplectic Lie algebra $\mathfrak{sp}(TM)$ is **naturally identified with the space $\Lambda^3(M)$** **(this is proven later today).**

Symplectic connections (reminder)

DEFINITION: When $B = \Lambda^1 M$, consider the exterior multiplication map $\operatorname{Alt}: \Lambda^i M \otimes \Lambda^1 M \longrightarrow \Lambda^{i+1} M$. Define **the torsion map** $T_\nabla(\eta) := \operatorname{Alt}(\nabla(\eta)) - d\eta$. Then T_∇ is equal to torsion on $\Lambda^1 M$ and satisfies the Leibnitz identity:

$$T_\nabla(\lambda \wedge \mu) = T_\nabla(\lambda) \wedge \mu + (-1)^{\tilde{\lambda}} \lambda \wedge T_\nabla(\mu) \qquad (**)$$

DEFINITION: An almost symplectic structure on a manifold is a non-degenerate 2-form.

EXERCISE: Let (M, ω) be an almost symplectic manifold. Prove that there exists a connection ∇ on TM such that $\nabla(\omega) = 0$. We call such connection **a symplectic connection**.

Lemma 1: Let $\omega \in \Lambda^2 M$ be an almost symplectic structure, and ∇ a symplectic connection. Using ω, we will identify TM and $\Lambda^1 M$, and then we can consider the torsion tensor $\mathfrak{T} \in \Lambda^2 M \otimes TM$ of ∇ as $\tau \in \Lambda^2 M \otimes \Lambda^1 M$. Let $\rho := \operatorname{Alt}(\tau)$. **Then $d\omega = -2\rho$.**

Proof: Clearly, $T_\nabla(\omega) = -d\omega$, because $\nabla(\omega) = 0$ and $T_\nabla(\omega) = \operatorname{Alt}(\nabla(\omega)) - d\omega$. By (**), we have $T_\nabla(\omega) = \operatorname{Alt}(A_1(\omega \otimes \mathfrak{T}) - A_2(\omega \otimes \mathfrak{T}))$, where A_i is the convolution of i-th component of $\omega \otimes T_\nabla$ and the last, taking $\Lambda^2 M \otimes \Lambda^2 M \otimes TM$ to $\Lambda^2 M \otimes \Lambda^1 M$ and $\Lambda^1 M \otimes \Lambda^2 M$. Clearly, $\operatorname{Alt}(A_1(\omega \otimes T_\nabla)) = -\operatorname{Alt}(A_2(\omega \otimes T_\nabla)) = \rho$. This gives $T_\nabla(\omega) = d\omega = -2\rho$. ■

Torsion of almost symplectic structures (reminder)

Theorem 1: Let (M, ω) be an almost symplectic manifold, and ∇ a symplectic connection. Denote its torsion by $T_\nabla \in \Lambda^2 M \otimes TM$. Using the form ω, we identify TM and $\Lambda^1 M$ and consider T_∇ as a section $\tau \in \Lambda^2 M \otimes \Lambda^1 M$. Denote by Alt_{123} the multiplication map $\Lambda^2 M \otimes \Lambda^1 M \longrightarrow \Lambda^3 M$. **Then $\operatorname{Alt}_{123}(\tau) = -\frac{1}{2} d\omega$.** Moreover, **any tensor $\mathfrak{T} \in \Lambda^2 M \otimes \Lambda^1 M$ such that $\operatorname{Alt}_{123}(\mathfrak{T}) = -\frac{1}{2} d\omega$ can be realized as a torsion of a symplectic connection.**

Proof. Step 1: Let $\mathfrak{sp}(TM)$ be the Lie algebra of all tensors $a \in \operatorname{End}(TM)$ such that $\omega(a(x), y) = -\omega(x, a(y))$. The same argument as the one used to show $\mathfrak{so}(TM) = \Lambda^2 M$ shows that $\mathfrak{sp}(TM) = \operatorname{Sym}^2(\Lambda^1 M)$.

Step 2: Under this identification, the linearized torsion map $T_{\text{lin}} : \Lambda^1 M \otimes \mathfrak{sp}(TM) \longrightarrow \Lambda^2 M \otimes TM$ becomes $\operatorname{Alt}_{12} : \Lambda^1 M \otimes \operatorname{Sym}^2(\Lambda^1 M) \longrightarrow \Lambda^2 M \otimes \Lambda^1 M$. Kernel of this map is clearly $\operatorname{Sym}^3(\Lambda^1 M)$. This gives an exact sequence **(check it).**

$$0 \longrightarrow \operatorname{Sym}^3(\Lambda^1 M) \hookrightarrow \Lambda^1 M \otimes \operatorname{Sym}^2(\Lambda^1 M) \xrightarrow{\operatorname{Alt}_{12}} \Lambda^2 M \otimes \Lambda^1 M \xrightarrow{\operatorname{Alt}_{123}} \Lambda^3 M \longrightarrow 0.$$

We identified $\Lambda^3 M$ with the space of intrinsic torsion for $\mathfrak{sp}(TM)$.

Step 3: $\operatorname{Alt}_{123}(\tau) = -\frac{1}{2} d\omega$ (Lemma 1). This is precisely the intrinsic torsion of ∇. ■

Torsion of unitary connection on a complex manifold

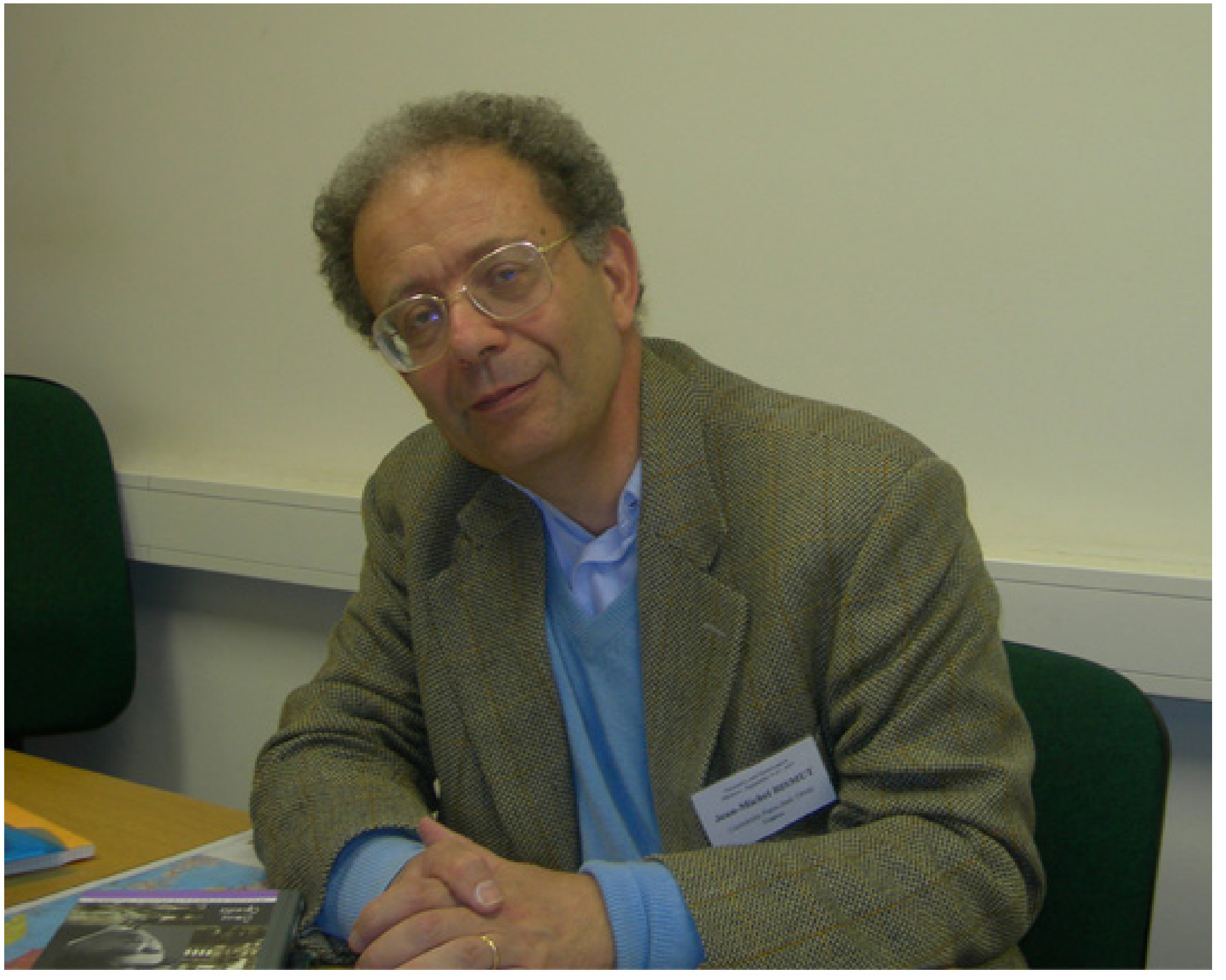
PROPOSITION: Let (M, I, ω) be an Hermitian complex manifold, ∇ a connection on TM preserving I and ω, and $T_\nabla \in \Lambda^2 M \otimes TM = \Lambda^2 M \otimes \Lambda^1 M$ (we identify TM and $\Lambda^1 M$ using the Riemannian structure). **Then**

$$T_\nabla \in \left(\Lambda^{2,0}(M) \otimes \Lambda^{0,1}(M)\right) \oplus \left(\Lambda^{0,2} \otimes \Lambda^{1,0}(M)\right) \oplus \left(\Lambda^{1,1}(M) \otimes \Lambda^1 M\right). \qquad (**)$$

Proof. Step 1: Integrability of I implies that $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$. Since $\nabla(I) = 0$, one also has $\nabla_X(T^{1,0}M) \subset T^{1,0}M$ for any vector field $X \in TM$. This gives $\nabla_X(Y) - \nabla_Y(X) - [X,Y] \in T^{1,0}M$ for any $X, Y \in T^{1,0}M$. We have shown that

$$T_\nabla \in \left(\Lambda^{2,0}(M) \otimes T^{1,0}(M)\right) \oplus \left(\Lambda^{0,2} \otimes T^{0,1}(M)\right) \oplus \left(\Lambda^{1,1}(M) \otimes \Lambda^1 M\right).$$

Step 2: Since the Riemannian form g is of type (1,1), it pairs (0,1)-vectors and (1,0)-vectors. Therefore, it identifies $T^{1,0}M$ with $\Lambda^{0,1}(M)$. This proves (**). ■



Jean-Michel Bismut (born 26 February 1948)

Bismut connection

THEOREM: (Bismut) Let (M, I, ω) be an Hermitian complex manifold. Then there exists a unique connection ∇ preserving I and ω, such that its torsion $T_\nabla \in \Lambda^2 M \otimes TM = \Lambda^2 M \otimes \Lambda^1 M$ (we identify TM and $\Lambda^1 M$ using the Riemannian metric) **is antisymmetric:** $T_\nabla \in \Lambda^3 M \subset \Lambda^2 M \otimes \Lambda^1 M$. Moreover, **in this case $T_\nabla = -\frac{1}{2} I(d\omega)$.**

REMARK: This connection is called **the Bismut connection**. When (M, I, ω) is Kähler, it is torsion-free and orthogonal, hence **∇ is the Levi-Civita connection.** We obtain that **on a Kähler manifold, Levi-Civita connection satisfies $\nabla(I) = 0$.**

Proof. Step 1: There are two different ways to identify $\Lambda^2 M \otimes TM$ and $\Lambda^2 M \otimes \Lambda^1 M$: using $g: TM \xrightarrow{\sim} \Lambda^1 M$ and using $\omega: TM \xrightarrow{\sim} \Lambda^1 M$. Denote the first tensor by τ_g and the second by τ_ω. It is clear that $I_3(\tau_g) = \tau_\omega$, where $I_3(x \otimes y \otimes z) = x \otimes y \otimes I(z)$. Torsion of symplectic connections was described earlier today (Theorem 1): we have shown that $\operatorname{Alt}(\tau_\omega) = -\frac{1}{2} d\omega$. **This implies that the image of the linearized torsion $T_{\operatorname{lin}}(\Lambda^1 M \otimes \mathfrak{u}(TM))$ satisfies $\operatorname{Alt}(I_3(T_{\operatorname{lin}}(\Lambda^1 M \otimes \mathfrak{u}(TM))) = 0$.** Indeed, $\operatorname{Alt}(I_3(T_\nabla))$ is independent from ∇ for any Hermitian connection ∇, **hence the linearization of the affine map $\nabla \mapsto \operatorname{Alt}(I_3(T_\nabla))$ vanishes.**

Bismut connection (2)

Proof. Step 1: The image of the linearized torsion $T_{\text{lin}}(\Lambda^1 M \otimes \mathfrak{u}(TM))$ **satisfies** $\operatorname{Alt}(I_3(T_{\text{lin}}(\Lambda^1 M \otimes \mathfrak{u}(TM)))) = 0$.

Step 2: The torsion of ∇ belongs to the space

$$\mathfrak{W} := \left(\Lambda^{2,0}(M) \otimes \Lambda^{0,1}(M)\right) \oplus \left(\Lambda^{0,2} \otimes \Lambda^{1,0}(M)\right) \oplus \left(\Lambda^{1,1}(M) \otimes \Lambda^1 M\right),$$

as shown above. The linearized torsion map is $T_{\text{lin}}: \ \Lambda^1 M \otimes \mathfrak{u}(TM) \longrightarrow \mathfrak{W}$. By the same argument as in the proof of existence of Levi-Civita connection, this map is injective. **This gives an exact sequence**

$$0 \longrightarrow \Lambda^1 M \otimes \mathfrak{u}(TM) \xrightarrow{T_{\text{lin}}} \mathfrak{W} \xrightarrow{I_3 \circ \operatorname{Alt}} \Lambda^{2,1}(M) \oplus \Lambda^{1,2}(M) \longrightarrow 0, \quad (***)$$

The last arrow of (***) is surjective because any (2,1)+(1,2)-form can be obtained as anti-symmetrization of $\alpha \in I_3(\mathfrak{W})$. The sequence (***) is exact in the middle term because dimension of the middle term is equal to sum of dimensions of the left and right terms.

Bismut connection (3)

Step 2: Let $\mathfrak{W} := (\Lambda^{2,0}(M)\otimes\Lambda^{0,1}(M))\oplus(\Lambda^{0,2}\otimes\Lambda^{1,0}(M))\oplus(\Lambda^{1,1}(M)\otimes\Lambda^1 M)$. Then **the sequence**

$$0 \longrightarrow \Lambda^1 M\otimes\mathfrak{u}(TM) \xrightarrow{T_{\mathrm{lin}}} \mathfrak{W} \xrightarrow{I_3\circ\mathrm{Alt}} \Lambda^{2,1}(M)\oplus\Lambda^{1,2}(M) \longrightarrow 0 \quad (***)$$

is exact.

Step 3: Let $\mathfrak{U}\subset\mathfrak{W}$ be a subspace consisting of all antisymmetric 3-forms, $\mathfrak{U} = \Lambda^{2,1}(M)\oplus\Lambda^{1,2}(M)$. Clearly, for any differential form η, one has $\mathrm{Alt}(I_3(\eta)) = W(\eta)$, where W is **the Weil operator** acting as $W(\eta)(x,y,z) = \eta(Ix,y,z) + \eta(x,Iy,z)+\eta(x,y,Iz)$. Then $\mathfrak{U} \xrightarrow{I_3\circ\mathrm{Alt}} \Lambda^{2,1}(M)\oplus\Lambda^{1,2}(M)$ is bijective. Therefore, **there exists a unique form $\sigma\in\mathfrak{U}$ such that $\mathrm{Alt}(I_3(\sigma)) = -\frac{1}{2}d\omega$.**

Step 4: Let ∇_0 be a connection on TM which satisfies $\nabla_0(g) = \nabla_0(I) = 0$ **(prove that it exists)**, and $\tau_g\in\mathfrak{W}$ its torsion. Then $\mathrm{Alt}(I_3(\tau_g)) = \mathrm{Alt}(I_3(\sigma)) = -\frac{1}{2}d\omega$ by Theorem 1. Therefore, **there exists a unique $A\in\Lambda^1 M\otimes\mathfrak{u}(TM)$ such that $T_{\mathrm{lin}}(A)+\tau_g=\sigma$, and the torsion of connection $\nabla := \nabla_0 + A$ is equal to σ.**

Step 5: Step 3 gives $\sigma = \frac{1}{2}W^{-1}(d\omega)$. However, $d\omega$ is (2,1)+(1,2)-form, and for such forms $W = I$, hence $\sigma = -\frac{1}{2}I(d\omega)$. ■

Levi-Civita connection on Kähler manifolds

When $d\omega = 0$, this immediately implies

THEOREM: On a Kähler manifold (M, I, g, ω), **the Levi-Civita connection ∇ satisfies $\nabla(I) = \nabla(\omega) = 0$.**

Indeed, the Bismut connection is torsion-free in this case, hence coincides with Levi-Civita.

Let us prove this theorem directly.

Proof. Step 1: Let ∇ be a unitary connection on M, that is, one which satisfies $\nabla(I) = \nabla(\omega) = 0$ **(prove that it exists).** There are two different ways to identify $\Lambda^2 M \otimes TM$ and $\Lambda^2 M \otimes \Lambda^1 M$: using $g : TM \xrightarrow{\sim} \Lambda^1 M$ and using $\omega : TM \xrightarrow{\sim} \Lambda^1 M$. Denote the first tensor by τ_g and the second by τ_ω. It is clear that $I_3(\tau_g) = \tau_\omega$, where $I_3(x \otimes y \otimes z) = x \otimes y \otimes I(z)$. Torsion of symplectic connections was described earlier today (Theorem 1): we have shown that $\operatorname{Alt}(\tau_\omega) = -\frac{1}{2} d\omega$. **This implies that $\operatorname{Alt}(I_3(T_\nabla)) = 0$.** Since this is true for any unitary connection, **one also has $\operatorname{Alt}(I_3(T_{\mathrm{lin}}(\Lambda^1 M \otimes \mathfrak{u}(TM))) = 0$.**

Levi-Civita connection on Kähler manifolds (2)

Proof. Step 1: We proved that $\mathrm{Alt}(I_3(T_\nabla)) = 0$**,** where I_3 is I acting on the third tensor component. **Moreover,** $\mathrm{Alt}(I_3(T_{\mathrm{lin}}(\Lambda^1 M \otimes \mathfrak{u}(TM))) = 0$**.**

Step 2: The torsion of ∇ belongs to the space

$$\mathfrak{W} := \left(\Lambda^{2,0}(M) \otimes \Lambda^{0,1}(M)\right) \oplus \left(\Lambda^{0,2} \otimes \Lambda^{1,0}(M)\right) \oplus \left(\Lambda^{1,1}(M) \otimes \Lambda^1 M\right),$$

as shown above. The linearized torsion map is $T_{\mathrm{lin}}: \ \Lambda^1 M \otimes \mathfrak{u}(TM) \longrightarrow \mathfrak{W}$. By the same argument as in the proof of existence of Levi-Civita connection, this map is injective. **This gives an exact sequence**

$$0 \longrightarrow \Lambda^1 M \otimes \mathfrak{u}(TM) \xrightarrow{T_{\mathrm{lin}}} \mathfrak{W} \xrightarrow{I_3 \circ \mathrm{Alt}} \Lambda^{2,1}(M) \oplus \Lambda^{1,2}(M) \longrightarrow 0, \qquad (***)$$

The last arrow of (***) is surjective because any (2,1)+(1,2)-form can be obtained as anti-symmetrization of $\alpha \in I_3(\mathfrak{W})$. The sequence (***) is exact in the middle term because dimension of the middle term is equal to sum of dimensions of the left and right terms.

Step 3: Now, T_∇ satisfies $\mathrm{Alt}(I_3(T_\nabla)) = 0$, hence belongs to the image of T_{lin}. **Therefore, the connection** $\nabla - T_{\mathrm{lin}}^{-1}(T_\nabla)$ **is a torsion-free unitary connection.** ■

Laplacian on differential forms

DEFINITION: Let V be a vector space. **A metric g on V induces a natural metric on each of its tensor spaces:** $g(x_1 \otimes x_2 \otimes ... \otimes x_k, x'_1 \otimes x'_2 \otimes ... \otimes x'_k) = g(x_1, x'_1)g(x_2, x'_2)...g(x_k, x'_k)$.

This gives a natural positive definite scalar product on differential forms over a Riemannian manifold (M, g): $g(\alpha, \beta) := \int_M g(\alpha, \beta) \operatorname{Vol}_M$

DEFINITION: Let M be a Riemannian manifold. **Laplacian on differential forms** is $\Delta := dd^* + d^*d$.

REMARK: **Laplacian is self-adjoint and positive definite:** $(\Delta x, x) = (dx, dx) + (d^*x, d^*x)$. Also, Δ commutes with d and d^*.

THEOREM: **(The main theorem of Hodge theory)**
There is an orthonormal basis in the Hilbert space $L^2(\Lambda^*(M))$ consisting of eigenvectors of Δ. Moreover, **each eigenspace is finitely-dimensional, and the eigenvalues convergeto zero.**

THEOREM: **("Elliptic regularity for Δ")**
Let $\alpha \in L^2(\Lambda^k(M))$ be an eigenvector of Δ. **Then α is a smooth k-form.**

Fritz Alexander Ernst Noether

(October 7, 1884 - September 10, 1941)

Emmy Noether und Fritz Noether, 1933

De Rham cohomology

DEFINITION: The space $H^i(M) := \frac{\ker d\big|_{\Lambda^i M}}{d(\Lambda^{i-1}M)}$ is called **the de Rham cohomology of M**.

DEFINITION: A form α is called **harmonic** if $\Delta(\alpha) = 0$.

REMARK: Let α be a harmonic form. **Then** $(\Delta x, x) = (dx, dx) + (d^* x, d^* x)$, hence $\alpha \in \ker d \cap \ker d^*$.

REMARK: The projection $\mathcal{H}^i(M) \longrightarrow H^i(M)$ from harmonic forms to cohomology is injective. Indeed, a form α lies in the kernel of such projection if $\alpha = d\beta$, but then $(\alpha, \alpha) = (\alpha, d\beta) = (d^*\alpha, \beta) = 0$.

THEOREM: The natural map $\mathcal{H}^i(M) \longrightarrow H^i(M)$ is an isomorphism (see the next page).

REMARK: Poincare duality immediately follows from this theorem.

Hodge theory and the cohomology

THEOREM: The natural map $\mathcal{H}^i(M) \longrightarrow H^i(M)$ is an isomorphism.

Proof. Step 1: Since $d^2 = 0$ and $(d^*)^2 = 0$, one has $[d, \Delta] = dd^*d - dd^*d = 0$. This means that **$\Delta$ commutes with the de Rham differential.**

Step 2: Consider the eigenspace decomposition $\Lambda^*(M) \hat{\cong} \bigoplus_\alpha \Lambda^*_\alpha(M)$, where α runs through all eigenvalues of Δ, and $\Lambda^*_\alpha(M)$ is the corresponding eigenspace. **For each α, de Rham differential defines a complex**

$$\Lambda^0_\alpha(M) \xrightarrow{d} \Lambda^1_\alpha(M) \xrightarrow{d} \Lambda^2_\alpha(M) \xrightarrow{d} \dots$$

Step 3: On $\Lambda^*_\alpha(M)$, one has $dd^* + d^*d = \alpha$. When $\alpha \neq 0$, and η closed, this implies $dd^*(\eta) + d^*d(\eta) = dd^*\eta = \alpha\eta$, hence $\eta = d\xi$, with $\xi := \alpha^{-1}d^*\eta$. This implies that **the complexes $(\Lambda^*_\alpha(M), d)$ don't contribute to cohomology.**

Step 4: We have proven that

$$H^*(\Lambda^* M, d) = \bigoplus_\alpha H^*(\Lambda^*_\alpha(M), d) = H^*(\Lambda^*_0(M), d) = \mathcal{H}^*(M).$$

■