

Complex geometry

lecture 11: Supersymmetry for Kähler manifolds: Hodge $*$ operator

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Hodge * operator

Let V be a vector space. **A metric g on V induces a natural metric on each of its tensor spaces:** $g(x_1 \otimes x_2 \otimes \dots \otimes x_k, x'_1 \otimes x'_2 \otimes \dots \otimes x'_k) = g(x_1, x'_1)g(x_2, x'_2)\dots g(x_k, x'_k)$.

This gives a natural positive definite scalar product on differential k -forms over a Riemannian manifold (M, g) : $g(\alpha, \beta) := \sqrt{k!} \int_M g(\alpha, \beta) \text{Vol}_M$. We rescale it in such a way that the standard monomials become orthonormal.

Another non-degenerate form is provided by the **Poincare pairing**:

$$\alpha, \beta \longrightarrow \int_M \alpha \wedge \beta.$$

DEFINITION: Let M be a Riemannian n -manifold. Define **the Hodge * operator** $*$: $\Lambda^k M \longrightarrow \Lambda^{n-k} M$ by the following relation: $g(\alpha, \beta) = \int_M \alpha \wedge * \beta$.

REMARK: **The Hodge * operator always exists.** It is defined explicitly in an orthonormal basis $\xi_1, \dots, \xi_n \in \Lambda^1 M$:

$$*(\xi_{i_1} \wedge \xi_{i_2} \wedge \dots \wedge \xi_{i_k}) = (-1)^s \xi_{j_1} \wedge \xi_{j_2} \wedge \dots \wedge \xi_{j_{n-k}},$$

where $\xi_{j_1}, \xi_{j_2}, \dots, \xi_{j_{n-k}}$ is a complementary set of vectors to $\xi_{i_1}, \xi_{i_2}, \dots, \xi_{i_k}$, and s the signature of a permutation $(i_1, \dots, i_k, j_1, \dots, j_{n-k})$.

REMARK: $*^2|_{\Lambda^k(M)} = (-1)^{k(n-k)} \text{Id}_{\Lambda^k(M)}$

$$d^* = (-1)^{nk} * d*$$

CLAIM: On a compact Riemannian n -manifold, one has $d^*|_{\Lambda^k M} = (-1)^{nk} * d*$, where d^* denotes **the adjoint operator**, which is defined by the equation $(d\alpha, \gamma) = (\alpha, d^*\gamma)$.

Proof: Since

$$0 = \int_M d(\alpha \wedge \beta) = \int_M d(\alpha) \wedge \beta + (-1)^{\tilde{\alpha}} \alpha \wedge d(\beta),$$

one has $(d\alpha, * \beta) = (-1)^{\tilde{\alpha}} (\alpha, * d\beta)$. Setting $\gamma := * \beta$, we obtain

$$(d\alpha, \gamma) = (-1)^{\tilde{\alpha}} (\alpha, * d(*)^{-1} \gamma) = (-1)^{\tilde{\alpha}} (-1)^{\tilde{\alpha}(\tilde{n} - \tilde{\alpha})} (\alpha, * d* \gamma) = (-1)^{\tilde{\alpha}\tilde{n}} (\alpha, * d* \gamma).$$

■

REMARK: Since in all applications which we consider, n is even, **I would from now on ignore the sign $(-1)^{nk}$.**

Graded vector spaces and algebras (reminder)

DEFINITION: A **graded vector space** is a space $V^* = \bigoplus_{i \in \mathbb{Z}} V^i$.

REMARK: If V^* is graded, the endomorphisms space $\text{End}(V^*) = \bigoplus_{i \in \mathbb{Z}} \text{End}^i(V^*)$ is also graded, with $\text{End}^i(V^*) = \bigoplus_{j \in \mathbb{Z}} \text{Hom}(V^j, V^{i+j})$

DEFINITION: A **graded algebra** (or “graded associative algebra”) is an associative algebra $A^* = \bigoplus_{i \in \mathbb{Z}} A^i$, with the product compatible with the grading: $A^i \cdot A^j \subset A^{i+j}$.

REMARK: A bilinear map of graded spaces which satisfies $A^i \cdot A^j \subset A^{i+j}$ is called **graded**, or **compatible with grading**.

REMARK: The category of graded spaces can be defined as a **category of vector spaces with $U(1)$ -action**, with the weight decomposition providing the grading. Then **a graded algebra is an associative algebra in the category of spaces with $U(1)$ -action**.

DEFINITION: An operator on a graded vector space is called **even (odd)** if it shifts the grading by even (odd) number. The **parity** \tilde{a} of an operator a is 0 if it is even, 1 if it is odd. We say that an operator is **pure** if it is even or odd.

Supercommutator

DEFINITION: A **supercommutator** of pure operators on a graded vector space is defined by a formula $\{a, b\} = ab - (-1)^{\tilde{a}\tilde{b}}ba$.

DEFINITION: A graded associative algebra is called **graded commutative** (or “supercommutative”) if its supercommutator vanishes.

EXAMPLE: The Grassmann algebra is supercommutative.

DEFINITION: A **graded Lie algebra** (Lie superalgebra) is a graded vector space \mathfrak{g}^* equipped with a bilinear graded map $\{\cdot, \cdot\} : \mathfrak{g}^* \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ which is graded anticommutative: $\{a, b\} = -(-1)^{\tilde{a}\tilde{b}}\{b, a\}$ and satisfies **the super Jacobi identity** $\{c, \{a, b\}\} = \{\{c, a\}, b\} + (-1)^{\tilde{a}\tilde{c}}\{a, \{c, b\}\}$

EXAMPLE: Consider the algebra $\text{End}(A^*)$ of operators on a graded vector space, with supercommutator as above. **Then $\text{End}(A^*), \{\cdot, \cdot\}$ is a graded Lie algebra.**

Lemma 1: Let d be an odd element of a Lie superalgebra, satisfying $\{d, d\} = 0$, and L an even or odd element. **Then $\{\{L, d\}, d\} = 0$.**

Proof: $0 = \{L, \{d, d\}\} = \{\{L, d\}, d\} + (-1)^{\tilde{L}}\{d, \{L, d\}\} = 2\{\{L, d\}, d\}$. ■

Supersymmetry in Kähler geometry

Let (M, I, g) be a Kähler manifold, ω its Kähler form. **On $\Lambda^*(M)$, the following operators are defined.**

0. d, d^*, Δ , because it is Riemannian.
1. $L(\alpha) := \omega \wedge \alpha$
2. $\Lambda(\alpha) := *L*\alpha$. It is easily seen that $\Lambda = L^*$.
3. The Weil operator $\mathcal{W}|_{\Lambda^{p,q}(M)} = \sqrt{-1} (p - q)$

THEOREM: These operators generate a Lie superalgebra \mathfrak{a} of dimension $(5|4)$, acting on $\Lambda^*(M)$. Moreover, the Laplacian Δ is central in \mathfrak{a} , hence \mathfrak{a} also acts on the cohomology of M .

REMARK: This is a convenient way to summarize the Kähler relations and the Lefschetz' $\mathfrak{sl}(2)$ -action.

The coordinate operators

Let V be an even-dimensional real vector space equipped with a scalar product, and v_1, \dots, v_{2n} an orthonormal basis. Denote by $e_{v_i} : \Lambda^k V \rightarrow \Lambda^{k+1} V$ an operator of multiplication, $e_{v_i}(\eta) = v_i \wedge \eta$. Let $i_{v_i} : \Lambda^k V \rightarrow \Lambda^{k-1} V$ be an adjoint operator, $i_{v_i} = *e_{v_i}*$.

CLAIM: The operators $e_{v_i}, i_{v_i}, \text{Id}$ are a basis of an **odd Heisenberg Lie superalgebra** \mathfrak{H} , with **the only non-trivial supercommutator given by the formula** $\{e_{v_i}, i_{v_j}\} = \delta_{i,j} \text{Id}$.

Now, consider the tensor $\omega = \sum_{i=1}^n v_{2i-1} \wedge v_{2i}$, and let $L(\alpha) = \omega \wedge \alpha$, and $\Lambda := L^*$ be the corresponding **Hodge operators**.

CLAIM: (Lefschetz triples) From the commutator relations in \mathfrak{H} , one obtains immediately that

$$H := [L, \Lambda] = \left[\sum e_{v_{2i-1}} e_{v_{2i}}, \sum i_{v_{2i-1}} i_{v_{2i}} \right] = \sum_{i=1}^{2n} e_{v_i} i_{v_i} - \sum_{i=1}^{2n} i_{v_i} e_{v_i},$$

is a scalar operator acting as $k - n$ on k -forms.

COROLLARY: The triple L, Λ, H satisfies the relations for the $\mathfrak{sl}(2)$ Lie algebra: $[L, \Lambda] = H$, $[H, L] = 2L$, $[H, \Lambda] = 2\Lambda$.