Complex geometry

lecture 12: Kodaira relations

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Supercommutator (reminder)

DEFINITION: A **supercommutator** of pure operators on a graded vector space is defined by a formula $\{a,b\} = ab - (-1)^{\tilde a \tilde b} ba$.

DEFINITION: A graded associative algebra is called **graded commutative** (or "supercommutative") if its supercommutator vanishes.

EXAMPLE: **The Grassmann algebra is supercommutative.**

DEFINITION: **A graded Lie algebra** (Lie superalgebra) is a graded vector space $\mathfrak{g}^*$ equipped with a bilinear graded map $\{\cdot,\cdot\}: \mathfrak{g}^* \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ which is graded anticommutative: $\{a,b\} = -(-1)^{\tilde a \tilde b}\{b,a\}$ and satisfies **the super Jacobi identity** $\{c,\{a,b\}\} = \{\{c,a\},b\} + (-1)^{\tilde a \tilde c}\{a,\{c,b\}\}$

EXAMPLE: Consider the algebra $\operatorname{End}(A^*)$ of operators on a graded vector space, with supercommutator as above. **Then $\operatorname{End}(A^*), \{\cdot,\cdot\}$ is a graded Lie algebra.**

Lemma 1: Let d be an odd element of a Lie superalgebra, satisfying $\{d,d\} = 0$, and L an even or odd element. **Then $\{\{L,d\},d\} = 0$.**

Proof: $0 = \{L,\{d,d\}\} = \{\{L,d\},d\} + (-1)^{\tilde L}\{d,\{L,d\}\} = 2\{\{L,d\},d\}$. ■

Supersymmetry in Kähler geometry

Let (M, I, g) be a Kaehler manifold, ω its Kaehler form. **On $\Lambda^*(M)$, the following operators are defined.**

0. d, $d^* = *d*$, $\Delta = dd^* + d^*d$, because it is Riemannian.

1. $L(\alpha) := \omega \wedge \alpha$

2. $\Lambda(\alpha) := *L * \alpha$. It is easily seen that $\Lambda = L^*$.

3. The Weil operator $\mathcal{W}\Big|_{\Lambda^{p,q}(M)} = \sqrt{-1}\,(p-q)$

THEOREM: These operators generate a Lie superalgebra $\mathfrak{a}$ of dimension (5|4)**,** acting on $\Lambda^*(M)$. Moreover, the Laplacian Δ is central in $\mathfrak{a}$, hence **$\mathfrak{a}$ also acts on the cohomology of M.**

REMARK: This is a convenient way to summarize the Kähler relations and the Lefschetz' $\mathfrak{sl}(2)$-action.

Lefschetz triples (reminder)

Let V be an even-dimensional real vector space equipped with a scalar product, and $v_1, ..., v_{2n}$ an orthonormal basis. Denote by $e_{v_i}: \Lambda^k V \longrightarrow \Lambda^{k+1} V$ an operator of multiplication, $e_{v_i}(\eta) = v_i \wedge \eta$. Let $i_{v_i}: \Lambda^k V \longrightarrow \Lambda^{k-1} V$ be an adjoint operator, $i_{v_i} = *e_{v_i}*$.

CLAIM: The operators e_{v_i}, i_{v_i}, Id are a basis of an **odd Heisenberg Lie superalgebra** $\mathfrak{H}$, with **the only non-trivial supercommutator given by the formula** $\{e_{v_i}, i_{v_j}\} = \delta_{i,j}\,\mathrm{Id}$**.**

Now, consider the tensor $\omega = \sum_{i=1}^n v_{2i-1} \wedge v_{2i}$, and let $L(\alpha) = \omega \wedge \alpha$, and $\Lambda := L^*$ be the corresponding **Hodge operators**.

CLAIM: (Lefschetz triples) From the commutator relations in $\mathfrak{H}$, one obtains immediately that

$$H := [L, \Lambda] = \left[\sum e_{v_{2i-1}} e_{v_{2i}}, \sum i_{v_{2i-1}} i_{v_{2i}}\right] = \sum_{i=1}^{2n} e_{v_i} i_{v_i} - \sum_{i=1}^{2n} i_{v_i} e_{v_i},$$

is a scalar operator acting as $k-n$ on k-forms.

COROLLARY: The triple L, Λ, H satisfies the relations for the $\mathfrak{sl}(2)$ Lie algebra: $[L, \Lambda] = H$, $[H, L] = 2L$, $[H, \Lambda] = 2\Lambda$.

Hodge components of d (reminder)

CLAIM: Let (M,I) be an almost complex manifold, and $d=\oplus d^{i,1-i}$ be the Hodge components of d, with $d^{a,b}: \Lambda^{p,q}(M) \longrightarrow \Lambda^{p+a,q+b}(M)$. **Then there are only 4 components, $d=d^{2,-1}+d^{1,0},+d^{0,1}+d^{-1,2}$, with $d^{2,-1}$ and $d^{-1,2}$ C^∞-linear.** Moreover, **the operators $d^{-1,2}$ and $d^{2,-1}$ vanish when I is (formally) integrable.**

DEFINITION: The **twisted differential** is defined as $d^c := IdI^{-1}$.

CLAIM: Let (M,I) be a complex manifold. **Then $\partial := \frac{d-\sqrt{-1}\,d^c}{2}$, $\overline{\partial} := \frac{d+\sqrt{-1}\,d^c}{2}$ are the Hodge components of d,** $\partial = d^{1,0}$, $\overline{\partial} = d^{0,1}$.

Proof: The Hodge components of d are expressed as $d^{1,0}=\frac{d+\sqrt{-1}\,d^c}{2}$, $d^{0,1}=\frac{d-\sqrt{-1}\,d^c}{2}$. Indeed, $I(\frac{d-\sqrt{-1}\,d^c}{2})I^{-1}=\sqrt{-1}\frac{d-\sqrt{-1}\,d^c}{2}$, hence $\frac{d+\sqrt{-1}\,d^c}{2}$ **has Hodge type (1,0);** the same argument works for $\overline{\partial}$. ■

CLAIM: On a complex manifold, one has $d^c = [W, d]$**.**

Proof: Clearly, $[W, d^{1,0}]=\sqrt{-1}\,d^{1,0}$ and $[W, d^{0,1}]=-\sqrt{-1}\,d^{0,1}$. Then $[W,d]=\sqrt{-1}\,d^{1,0}-\sqrt{-1}\,d^{0,1}=IdI^{-1}$. ■

COROLLARY: $\{d,d^c\}=\{d,\{d,W\}\}=0$ (Lemma 1).

Plurilaplacian (reminder)

THEOREM: Let M, I be a complex manifold. **Then 1. $\partial^2 = 0$.**
2. $\overline{\partial}^2 = 0$.
3. $dd^c = -d^c d$
4. $dd^c = 2\sqrt{-1}\,\partial\overline{\partial}$.

Proof: The first is vanishing of (2,0)-part of d^2, and the second is vanishing of its (0,2)-part. Now, $\{d, d^c\} = -\{d, \{d, \mathcal{W}\}\} = 0$ (Lemma 1), this gives $dd^c = -d^c d$. Finally, $2\sqrt{-1}\partial\overline{\partial} = \frac{1}{2}(d + \sqrt{-1}\,d^c)(d - \sqrt{-1}\,d^c) = \frac{1}{2}(dd^c - d^c d) = dd^c$.
■

DEFINITION: The operator dd^c is called **the pluri-Laplacian**.

EXERCISE: Prove that **on a Riemannian surface (M, I, ω), one has $dd^c(f) = \Delta(f)\omega$.**

Kodaira identities

THEOREM: Let M be a Kaehler manifold. One has the following identities **("Kähler idenitities", "Kodaira idenities").**

$$[\Lambda, \partial] = \sqrt{-1}\,\overline{\partial}^*, \quad [L, \overline{\partial}] = -\sqrt{-1}\,\partial^*, \quad [\Lambda, \overline{\partial}^*] = -\sqrt{-1}\,\partial, \quad [L, \partial^*] = \sqrt{-1}\,\overline{\partial}.$$

Equivalently,

$$[\Lambda, d] = (d^c)^*, \qquad [L, d^*] = -d^c, \qquad [\Lambda, d^c] = -d^*, \qquad [L, (d^c)^*] = d.$$

Proof. Step 1: The first set of identities implies the second set. Indeed, by adding up appropriate identities in the top set of their complex conjugate, we obtain ones in the bottom set; for example, adding $[\Lambda, \partial] = \sqrt{-1}\,\overline{\partial}^*$ and $[\Lambda, \overline{\partial}] = -\sqrt{-1}\,\partial^*$,, we obtain $[\Lambda, d] = (d^c)^*$. Each of top identities is related to the other three by complex conjugation or by Hermitian conjugation, hence they are all equivalent. Each of the bottom identities implies the rest by Hermitian conjugation and conjugating with I. Finally, $[\Lambda, \partial] = \sqrt{-1}\,\overline{\partial}^*$ can be obtained as a sum of $[\Lambda, d] = (d^c)^*$ and $[\Lambda, d^c] = -d^*$ with appropriate coefficients. **We obtained that all Kodaira identities are implied by just one, say,** $[L, d^*] = -d^c$.

Kodaira identities (2)

Proof. Step 1: We reduced the Kodaira identities to just one, $[L, d^*] = -d^c$.

Step 2: Let $\mathfrak{E}: \Lambda^i M \otimes \Lambda^1 M \longrightarrow \Lambda^{i+1}(M)$ be the multiplication, and $\mathfrak{I}: \Lambda^i M \otimes \Lambda^1 M \longrightarrow \Lambda^{i-1}(M)$ the map that takes $\alpha \wedge \theta$ and puts it to $*(*\alpha \wedge \theta)$. In other words, $\mathfrak{I}$ takes a tensor $\alpha \otimes \theta$, with $\alpha \in \Lambda^i M$ and $\theta \in \Lambda^1 M$, uses the metric g to produce a vector field X from θ, and maps α to $\alpha \lrcorner X$ (convution of α and X).

Step 3: Let ∇ be the Levi-Civita connection. Then $d\alpha = \mathfrak{E}(\nabla(\alpha))$, because ∇ is torsion-free. Since $d^* = *d*$, one has $d^*(\alpha) = \mathfrak{I}(\nabla(\alpha))$. Let $x_1, y_1, ..., x_n, y_n \in \Lambda^1_m M$ be an orthonormal basis such that $\omega = \sum x_i \wedge y_i$. Then $\mathfrak{I}(\nabla(\alpha)) = \sum_i i_{x_i}(\nabla_{x_i}\alpha) + i_{y_i}(\nabla_{y_i}\alpha)$. Taking a commutator with $L = \sum e_{x_i} e_{y_i}$ and using the commutator relations between e_v and i_w found earlier, we obtain

$$[L, d^*] = \sum_i \nabla_{x_i}[e_{x_i}e_{y_i}, i_{x_i}] + \nabla_{y_i}[e_{x_i}e_{y_i}, i_{y_i}] = \sum_i \nabla_{y_i} e_{x_i} - \nabla_{x_i} e_{y_i}.$$

(the operator ∇_w commutes with L, because ω is parallel). However,

$$\sum_i \nabla_{y_i} e_{x_i} - \nabla_{x_i} e_{y_i} = -I\left(\sum_i \nabla_{x_i} e_{x_i} + \nabla_{y_i} e_{y_i}\right) = -d^c$$

which gives $[L, d^*] = -d^c$. ■

Laplacians and supercommutators

THEOREM: Let

$$\Delta_d := \{d, d^*\}, \quad \Delta_{d^c} := \{d^c, d^{c*}\}, \quad \Delta_\partial := \{\partial, \partial^*\}, \Delta_{\overline{\partial}} := \{\overline{\partial}, \overline{\partial}^*\}.$$

Then $\Delta_d = \Delta_{d^c} = 2\Delta_\partial = 2\Delta_{\overline{\partial}}$. In particular, **$\Delta_d$ preserves the Hodge decomposition.**

Proof: By Kodaira relations, $\{d, d^c\} = 0$. Graded Jacobi identity gives

$$\{d, d^*\} = -\{d, \{\Lambda, d^c\}\} = \{\{\Lambda, d\}, d^c\} = \{d^c, d^{c*}\}.$$

Same calculation with $\partial, \overline{\partial}$ gives $\Delta_\partial = \Delta_{\overline{\partial}}$. Also, $\{\partial, \overline{\partial}^*\} = \sqrt{-1}\{\partial, \{\Lambda, \partial\}\} = 0$, (Lemma 1), and the same argument implies that **all anticommutators $\partial, \overline{\partial}^*$, etc. all vanish except $\{\partial, \partial^*\}$ and $\{\overline{\partial}, \overline{\partial}^*\}$.** This gives $\Delta_d = \Delta_\partial + \Delta_{\overline{\partial}}$. ■

DEFINITION: The operator $\Delta := \Delta_d$ is called **the Laplacian**.

REMARK: We have proved that **operators $L, \Lambda, d, \mathscr{W}$ generate a Lie superalgebra of dimension $(5|4)$ (5 even, 4 odd), with $\mathbb{R}\Delta$ central.**

The Lefschetz $\mathfrak{sl}(2)$-action

COROLLARY: The operators L, Λ, H form a basis of a Lie algebra isomorphic to $\mathfrak{sl}(2)$, with relations

$$[L,\Lambda] = H, \quad [H,L] = 2L, \quad [H,\Lambda] = -2\Lambda.$$

DEFINITION: L, Λ, H is called **the Lefschetz $\mathfrak{sl}(2)$-triple**.

REMARK: Finite-dimensional representations of $\mathfrak{sl}(2)$ are semisimple.

REMARK: A simple finite-dimensional representation V of $\mathfrak{sl}(2)$ is generated by $v \in V$ which satisfies $\Lambda(v) = 0$, $H(v) = pv$ **("lowest weight vector")**, where $p \in \mathbb{Z}^{\geqslant 0}$. Then $v, L(v), L^2(v), ..., L^p(v)$ form a basis of $V_p := V$. **This representation is determined uniquely by p.**

REMARK: In this basis, H **acts diagonally:** $H(L^i(v)) = (2i - p)L^i(v)$.

REMARK: One has $V_p = \operatorname{Sym}^p V_1$, where V_1 is a 2-dimensional tautological representation. It is called **a weight p representation of $\mathfrak{sl}(2)$**.

COROLLARY: For a finite-dimensional representation V of $\mathfrak{sl}(2)$, denote by $V^{(i)}$ the eigenspaces of H, with $H\Big|_{V^{(i)}} = i$. **Then L^i induces an isomorphism $V^{(-i)} \xrightarrow{L^i} V^{(i)}$ for any $i > 0$.**

Lefschetz action on cohomology.

From the supersymmetry theorem, the following result follows.

COROLLARY: The $\mathfrak{sl}(2)$-action $\langle L, \Lambda, H\rangle$ and the action of Weil operator commute with Laplacian, hence **preserve the harmonic forms on a Kähler manifold**.

COROLLARY: Any cohomology class can be represented as a sum of closed (p,q)-forms, giving a decomposition $H^i(M) = \bigoplus_{p+q=i} H^{p,q}(M)$, with $\overline{H^{p,q}(M)} = H^{q,p}(M)$.

COROLLARY: odd cohomology of a compact Kähler manifold are even-dimensional.

COROLLARY: Let M be a compact, Kähler manifold of complex dimension n, and $i+p+q=n$. Then L^i defines **the Lefschetz isomorphism** $H^{p,q} \xrightarrow{L^i} H^{p+2i,q+2i}(M)$

The Hodge diamond:

$$
\begin{array}{ccccccc}
 & & & H^{n,n} & & & \\
 & & H^{n,n-1} & & H^{n-1,n} & & \\
 & H^{n,n-2} & & H^{n-1,n-1} & & H^{n-2,n} & \\
H^{n,n-3}(M) & & H^{n-1,n-2}(M) & & H^{n-2,n-1}(M) & & H^{n-3,n}(M) \\
\vdots & & \vdots & & \vdots & & \vdots \\
H^{3,0}(M) & & H^{2,1}(M) & & H^{1,2}(M) & & H^{0,3}(M) \\
 & H^{2,0} & & H^{1,1} & & H^{0,2} & \\
 & & H^{1,0} & & H^{0,1} & & \\
 & & & H^{0,0} & & &
\end{array}
$$