# Complex geometry

## lecture 13: Cohomology and a circle action

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## Stone-Weierstrass approximation theorem

**DEFINITION:** Let $M$ be a topological space, and $\|f\| := \sup_M |f|$ **the sup-norm on functions**. $C^0$**-topology** on the space $C^0(M)$ of continuous, bounded real-valued functions is the topology defined by the sup-norm.

**EXERCISE:** Prove that $C^0(M)$ **with sup-norm is a complete metric space.**

**DEFINITION:** Let $A \subset C^0 M$ be a subspace in the space of continuous functions. We say that $A$ **separates the points** of $M$ if for all distinct points $x, y \in M$, there exists $f \in A$ such that $f(x) \neq f(y)$.

**THEOREM: (Stone-Weierstrass theorem)**
Let $A \subset C^0 M$ be a subring separating points, and $\overline{A}$ its closure. **Then** $\overline{A} = C^0 M$**.**

## Hilbert spaces

**DEFINITION: Hilbert space** over $\mathbb{C}$ is a complete, infinite-dimensional Hermitian space which is second countable (that is, has a countable dense set).

**DEFINITION: Orthonormal basis** in a Hilbert space $H$ is a set of pairwise orthogonal vectors $\{x_\alpha\}$ which satisfy $|x_\alpha| = 1$, and such that $H$ is the closure of the subspace generated by the set $\{x_\alpha\}$.

**THEOREM: Any Hilbert space has a basis, and all such bases are countable.**

**Proof:** A basis is found using Zorn lemma. If it's not countable, open balls with centers in $x_\alpha$ and radius $\varepsilon < 2^{-1/2}$ don't intersect, which means that the second countability axiom is not satisfied. ■

**THEOREM: All Hilbert spaces are isometric**.

**Proof:** Each Hilbert space has a countable orthonormal basis. ■

## Fourier series

**EXAMPLE:** Let $(M, \mu)$ be a space with measure. Consider the space $V$ of measurable functions $f: M \longrightarrow \mathbb{C}$ such that $\int_M |f|^2\mu < \infty$. For each $f, g \in V$, the integral $\int f\bar{g}\mu$ is well defined, by Cauchy inequality: $\int |fg|\mu < \sqrt{\int_M |f|^2\mu \int_M |g|^2\mu}$. This gives a Hermitian form on $V$. Let $L^2(M)$ denote the completion of $V$ with respect to this metric. It is called **the space of square-integrable functions on $M$**. Its elements are called **$L^2$-functions.**

**CLAIM: ("Fourier series")** Functions $e_k(t) = e^{2\pi\sqrt{-1}\,kt}$, $k \in \mathbb{Z}$ on $S^1 = \mathbb{R}/\mathbb{Z}$ **form an orthonormal basis in the Hilbert space $L^2(S^1)$**.

**Proof. Step 1:** Orthogonality is clear from $\int_{S^1} e^{2\pi\sqrt{-1}\,kt}dt = 0$ for all $k \neq 0$ (prove it).

**Step 2:** The space of Fourier polynomials $\sum_{i=-n}^{n} a_k e_k(t)$ is dense in the space of continuous functions on the circle by the Stone-Weierstrass approximation theorem. Therefore, the closure of the space of functions which admit Fourier series is $L^2(S^1)$. ∎

## Fourier series on a torus

**REMARK:** Let $t_1, ..., t_n$ be coordinates on $\mathbb{R}^n$. We can think of $t_i$ as of angle coordinates on the torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$, considered as a product of $n$ copies of $S^1$. Consider the **Fourier monomials** $F_{l_1,...,l_n} := \exp(2\pi\sqrt{-1}\sum_{i=1}^n l_i t_i)$, where $l_1, ..., l_n$ are integers. Clearly,

$$L^2(T^n) \cong \underbrace{L^2(S^1)\hat{\otimes}L^2(S^1)\hat{\otimes}...\hat{\otimes}L^2(S^1)}_{n \text{ times}}.$$

where $\hat{\otimes}$ denotes the completed tensor product. This implies that the **Fourier monomials form a Hilbert basis in $L^2(T^n)$.**

**REMARK:** This also follows directly from the Stone-Weierstrass theorem.

**THEOREM:** Let $V$ be a Hilbert space, Map$(T^n, V)$ continuous maps, and $L^2(T^n, V)$ a completion of Map$(T^n, V)$ with respect to the $L^2$-norm $|v|^2 = \int_{T^n} |v(x)|^2 dx$. Consider an orthonormal basis $u_1, ..., u_n, ...$ in $V$. Then **an orthonormal basis in Map$(T^n, V)$ is given by monomial maps $F_{l_1,...,l_n} u_j$** taking $s \in T^n$ to $F_{l_1,...,l_n}(s) u_j$.

**Proof:** Orthonormality of the collection $\{F_{l_1,...,l_n} u_j\}$ is clear. To prove its completeness (that is, the density of the subspace generated by $\{F_{l_1,...,l_n} u_j\}$), notice that Map$(T^n, V)$ is a completion of $\oplus_i$ Map$(T^n, V_i)$, where $V_i = \langle v_i \rangle$. Now, $\{F_{l_1,...,l_n} u_i\}$ is an orthonormal basis in $V_i =$ Map$(T^n, \mathbb{C})$. ■

## Weight decomposition for $U(1)$-representations

**EXERCISE:** Let $\rho : U(1) \longrightarrow GL(V)$ be a finite-dimensional irreducible complex representation of the Lie group $U(1)$. **Prove that $\dim \mathbb{C} = 1$ and there exists $n \in \mathbb{Z}$ such that $t \in U(1) = \mathbb{R}/\mathbb{Z}$ acts on $V$ as $\rho(t)(v) = e^{2\pi\sqrt{-1}\,nt}v$.**

**DEFINITION:** A representation of $U(1)$ with $\rho(t)(v) = e^{2\pi\sqrt{-1}\,nt}v$ is called **weight $n$** representation.

**DEFINITION:** Let $V$ be a Hermitian space (possibly infinitely-dimensional) equipped with an action of $U(1)$, and $V_k \subset V$ weight $k$ representations, $k \in \mathbb{Z}$. The direct sum $\bigoplus V_k$ is called **the weight decomposition** for $V$ if it is dense in $V$.

**EXAMPLE:** Let $L^2(S^1, W)$ the space of maps from $S^1$ to a Hermitian space $W$. We define $U(1)$-action on $L^2(S^1, W)$ by $\rho(t)(f) = R_t(f)$ where $R_t(f(x)) = f(x+t)$ shifts $S^1$ by $t$. Clearly, this is a Hermitian representation, and **its weight decomposition is its Fourier decomposition.**

## Weight decomposition for $U(1)$-representations (2)

**CLAIM:** Let $\bigoplus V_k \subset V$ be the weight decomposition of a Hermitian representation $\rho$ of $U(1)$. Then **any vector $v \in V$ can be decomposed onto a converging serie $v = \sum_{i\in\mathbb{Z}} v_i$, with $v_i \in V_i$.** This decomposition is called **the weight decomposition** for $v$.

**Proof. Step 1:** Clearly, all $V_i$ are pairwise orthogonal; indeed, for any $t \in U(1)$ and $x_p \in V_p$, $x_q \in V_q$, $i \neq j$, we have

$$e^{2\pi\sqrt{-1}\,pt}(x_p, x_q) = (\rho(t)(x_p), x_q) = (x_p, \rho(-t)x_q) =$$
$$= (x_p, e^{-2\pi\sqrt{-1}\,qt}x_q) = e^{2\pi\sqrt{-1}\,qt}(x_p, x_q)$$

giving $p = q$ whenever $(x_p, x_q) \neq 0$.

**Step 2:** Let $\pi_i : V \longrightarrow V_i$ be the orthogonal projection. Then $|x|^2 \geqslant \sum_{i=-p}^{p} |\pi_i(x)|^2$ because orthogonal projection is always distance-decreasing. Therefore, the serie $\sum_{i\in\mathbb{Z}} \pi_i(x)$ converges. Its limit is a vector $x'$ which satisfies $(x, u) = (x', u)$ for any $u \in \bigoplus_{k\in\mathbb{Z}} V_k$. Since $\bigoplus_{k\in\mathbb{Z}} V_k$ is dense in $V$, this implies $x = x'$. ■

## Weight decomposition and Fourier series

**LEMMA:** Let $W$ be a Hermitian representation of $U(1)$ admitting a weight decomposition. Then **any subquotient of $W$ also admits a weight decomposition.**

**Proof:** This is clear for quotients. Any closed subspace $V \subset W$ gives a direct sum decomposition $W = V \oplus V^\perp$, hence it also can be realized as a quotient. ■

**LEMMA:** Let $\rho : U(1) \longrightarrow U(W)$ be a Hermitian representation of $U(1)$, and $L^2(S^1, W)$ the space of maps from $S^1$ to $W$ with the $U(1)$-action by translation as defined earlier. **Then $W$ can be realized as a sub-representation of $L^2(S^1, W)$.**

**Proof:** For any $x \in W$ consider $\alpha_x \in L^2(S^1, W)$ taking $t \in U(1) = \mathbb{R}/\mathbb{Z}$ to $\rho(t)(x)$. Clearly, $x \mapsto \alpha_x$ **defines a homomorphism of representations.** ■

**THEOREM:** Let $W$ be a Hermitian representation of $U(1)$. **Then $W$ admits a weight decomposition $W = \widehat{\bigoplus_{i\in\mathbb{Z}} W_i}$.**

**Proof:** We realize $W$ as a subrepresentation in $L^2(S^1, W)$, and use the Fourier series to obtain the weight decomposition of $L^2(S^1, W)$. ■

## Weight decomposition for $T^n$-action

**EXERCISE:** Consider the $n$-dimensional torus $T^n$ as a Lie group, $T^n = U(1)^n$. **Prove that any finite-dimensional Hermitian representation of $T^n$ is a direct sum of 1-dimensional representations,** with action of $T^n$ given by $\rho(t_1, ..., t_n)(x) = \exp(2\pi\sqrt{-1}\ \sum_{i=1}^n p_i t_i)x$, for some $p_1, ..., p_n \in \mathbb{Z}^n$, called **the weights** of the 1-dimensional representation.

**DEFINITION:** Let $V$ be a Hermitian space (possibly infinitely-dimensional) equipped with an action of $T^n$, and $V_\alpha \subset V$ weight $\alpha$ representations, $\alpha \in \mathbb{Z}^n$. The direct sum $\bigoplus_{\alpha\in\mathbb{Z}^n} V_\alpha$ is called **the weight decomposition** for $V$ if it is dense in $V$.

**THEOREM:** Let $W$ be a Hermitian vector space. Then **the Fourier series provide the weight decomposition on $L^2(T^n, W)$.** ■

**THEOREM:** Let $W$ be a Hermitian representation of $T^n$. **Then $W$ admits a weight decomposition $V = \widehat{\bigoplus_{\alpha\in\mathbb{Z}^n} W_\alpha}$.**

**Proof:** We realize $W$ as a subrepresentation in $L^2(T^n, W)$, and use the Fourier series to obtain the weight decomposition of $L^2(T^n, W)$. ■

## Weight decomposition for $T^n$-action on differential forms

**REMARK:** Let $M$ be a manifold with the $T^n$-action, and

$$\Lambda^*(M) = \widehat{\bigoplus}_{\alpha\in\mathbb{Z}^n}\Lambda^*(M)_{p_1,\dots,p_k}$$

be the weight decomposition on the differential forms. Then the **de Rham differential preserves each term $\Lambda^*(M)_{p_1,\dots,p_k}$.** Indeed, $d$ **commutes with the action of the Lie algebra of $T^n$, and $\Lambda^*(M)_{p_1,\dots,p_k}$ are its eigenspaces.**

**REMARK:** The weight decomposition $\alpha = \sum \alpha_{p_1,\dots,p_k}$ converges, generally speaking, only in $L^2$, however, if action of $T^n$ is smooth, it converges uniformly in $t \in T^n$ because **the Fourier serie of a smooth function converge uniformly**.

**REMARK:** Let $\alpha = \sum \alpha_{p_1,\dots,p_k}$ be the weight decomposition. **The forms $\alpha_{p_1,\dots,p_k}$ are obtained by averaging**

$$e^{2\pi\sqrt{-1}\,\sum_{i=1}^n p_i t_i}\alpha = \operatorname{Av}_{T^n} e^{2\pi\sqrt{-1}\,\sum_{i=1}^n -p_i t_i}\alpha$$

**hence they are smooth.**

## De Rham cohomology and $T^n$-action

**THEOREM:** Let $M$ be a smooth manifold, and $T^n$ a torus acting on $M$ by diffeomorphisms. Denote by $\Lambda^*(M)^{T^n}$ the complex of $T^n$-invariant differential forms. **Then the natural embedding $\Lambda^*(M)^{T^n} \hookrightarrow \Lambda^*(M)$ induces an isomorphism on de Rham cohomology.**

**Proof. Step 1:** Let $\alpha \in \Lambda^*(M)$ be a form and $\alpha = \sum \alpha_{p_1,...,p_n}$ its weight decomposition, with $\alpha_{p_1,...,p_n} \in \Lambda^*_{p_1,...,p_n}(M)$ a form of weight $p_1, ..., p_n$. Since $T^n$-action commutes with de Rham differential, these forms are closed when $\alpha$ is closed.

**Step 2:** Let $r_1, ..., r_n$ be the standard generators of the Lie algebra of $T^n$ rescaled in such a way that $\operatorname{Lie}_{r_k}(\exp(2\pi\sqrt{-1}\ \sum_{i=1}^n p_i t_i)) = \sqrt{-1}\, p_k$, and $i_{r_k}: \Lambda^i(M) \longrightarrow \Lambda^{i-1}(M)$ the convolution operator. Since $\operatorname{Lie}_{r_k} = \{d, i_{r_k}\}$, we have $p_k \alpha_{p_1,...,p_n} = d(i_{r_k}\alpha_{p_1,...,p_n})$ whenever $\alpha_{p_1,...,p_n}$ is closed. Therefore, **all terms in the weight decomposition $\alpha = \sum \alpha_{p_1,...,p_n}$ are exact except $\alpha_{0,0,...,0}$.**

**Step 3:** In the direct sum decomposition of the de Rham complex

$$\Lambda^*(M) = \Lambda^*(M)^{T^n} \oplus \widehat{\bigoplus}_{p_1,...,p_k \neq (0,0,...,0)} \Lambda^*_{p_1,...,p_k}(M)$$

the second component has trivial cohomology, because $\operatorname{Lie}_{r_k}$ is invertible on $\bigoplus_{p_k \neq 0} \Lambda^*_{p_1,...,p_n}(M)$ **(deduce it from $p_k \alpha_{p_1,...,p_k} = d(i_{r_k}\alpha_{p_1,...,p_k})$)**, and $\operatorname{Lie}_{r_k}$(closed form) is exact. ■