Complex geometry

lecture 14: Dolbeault cohomology of a torus

Misha Verbitsky

HSE, room 306, 16:20,

November 7, 2020

Weight decomposition for T^n-action (reminder)

EXERCISE: Consider the n-dimensional torus T^n as a Lie group, $T^n = U(1)^n$. **Prove that any finite-dimensional Hermitian representation of T^n is a direct sum of 1-dimensional representations,** with action of T^n given by $\rho(t_1, ..., t_n)(x) = \exp(2\pi\sqrt{-1}\ \sum_{i=1}^n p_i t_i)x$, for some $p_1, ..., p_n \in \mathbb{Z}^n$, called **the weights** of the 1-dimensional representation.

DEFINITION: Let V be a Hermitian space (possibly infinitely-dimensional) equipped with an action of T^n, and $V_\alpha \subset V$ weight α representations, $\alpha \in \mathbb{Z}^n$. The direct sum $\bigoplus_{\alpha\in\mathbb{Z}^n} V_\alpha$ is called **the weight decomposition** for V if it is dense in V.

THEOREM: Let W be a Hermitian vector space. Then **the Fourier series provide the weight decomposition on $L^2(T^n, W)$.** ∎

THEOREM: Let W be a Hermitian representation of T^n. **Then W admits a weight decomposition $V = \widehat{\bigoplus_{\alpha\in\mathbb{Z}^n} W_\alpha}$.**

Proof: We realize W as a subrepresentation in $L^2(T^n, W)$, and use the Fourier series to obtain the weight decomposition of $L^2(T^n, W)$. ∎

Weight decomposition for T^n-action on differential forms (reminder)

REMARK: Let M be a manifold with the T^n-action, and

$$\Lambda^*(M) = \hat{\bigoplus}_{\alpha\in\mathbb{Z}^n}\Lambda^*(M)_{p_1,...,p_k}$$

be the weight decomposition on the differential forms. Then the **de Rham differential preserves each term $\Lambda^*(M)_{p_1,...,p_k}$.** Indeed, d **commutes with the action of the Lie algebra of T^n, and $\Lambda^*(M)_{p_1,...,p_k}$ are its eigenspaces.**

REMARK: Let $\alpha = \sum \alpha_{p_1,...,p_k}$ be the weight decomposition. **The forms $\alpha_{p_1,...,p_k}$ are obtained by averaging**

$$e^{2\pi\sqrt{-1}\sum_{i=1}^n p_i t_i}\alpha = \operatorname{Av}_{T^n} e^{2\pi\sqrt{-1}\sum_{i=1}^n -p_i t_i}\alpha$$

hence they are smooth.

De Rham cohomology and T^n-action (reminder)

THEOREM: Let M be a smooth manifold, and T^n a torus acting on M by diffeomorphisms. Denote by $\Lambda^*(M)^{T^n}$ the complex of T^n-invariant differential forms. **Then the natural embedding $\Lambda^*(M)^{T^n} \hookrightarrow \Lambda^*(M)$ induces an isomorphism on de Rham cohomology.**

Proof. Step 1: Let $\alpha \in \Lambda^*(M)$ be a form and $\alpha = \sum \alpha_{p_1,...,p_n}$ its weight decomposition, with $\alpha_{p_1,...,p_n} \in \Lambda^*_{p_1,...,p_n}(M)$ a form of weight $p_1,...,p_n$. Since T^n-action commutes with de Rham differential, these forms are closed when α is closed.

Step 2: Let $r_1,...,r_n$ be the standard generators of the Lie algebra of T^n rescaled in such a way that $\operatorname{Lie}_{r_k}(\exp(2\pi\sqrt{-1}\sum_{i=1}^n p_i t_i)) = \sqrt{-1}\, p_k$, and $i_{r_k}: \Lambda^i(M) \longrightarrow \Lambda^{i-1}(M)$ the convolution operator. Since $\operatorname{Lie}_{r_k} = \{d, i_{r_k}\}$, we have $p_k\alpha_{p_1,...,p_n} = d(i_{r_k}\alpha_{p_1,...,p_n})$ whenever $\alpha_{p_1,...,p_n}$ is closed. Therefore, **all terms in the weight decomposition $\alpha = \sum \alpha_{p_1,...,p_n}$ are exact except $\alpha_{0,0,...,0}$.**

Step 3: In the direct sum decomposition of the de Rham complex

$$\Lambda^*(M) = \Lambda^*(M)^{T^n} \oplus \widehat{\bigoplus}_{p_1,...,p_k \neq (0,0,...,0)} \Lambda^*_{p_1,...,p_k}(M)$$

the second component has trivial cohomology, because Lie_{r_k} is invertible on $\bigoplus_{p_k \neq 0} \Lambda^*_{p_1,...,p_n}(M)$ **(deduce it from $p_k\alpha_{p_1,...,p_k} = d(i_{r_k}\alpha_{p_1,...,p_k})$)**, and Lie_{r_k}(closed form) is exact. ∎

Constant forms on a torus

REMARK: In the proof above, the serie $\sum_{p_k\neq 0}\frac{1}{p_k} i_{r_k}\alpha_{p_1,\ldots,p_k}$ converges, because $\sum_{p_k\neq 0}\alpha_{p_1,\ldots,p_k}$ converges, and satisfies $d\left(\sum_{p_k\neq 0}\frac{1}{p_k} i_{r_k}\alpha_{p_1,\ldots,p_k}\right) = \sum_{p_k\neq 0}\alpha_{p_1,\ldots,p_k}$ as shown.

DEFINITION: Let $T^n = (S^1)^n$ be a compact torus equipped with a action on itself by shifts, and $\Lambda^*_{\text{const}}(M)$. the space of T^n-invariant forms on T^n. These forms are called **constant differential forms**. Clearly, **constant forms have constant coefficients in the usual (flat) coordinates on the torus.**

THEOREM: The natural embedding $\Lambda^*_{\text{const}}(T^n) \hookrightarrow \Lambda^*(T^n)$ **induces an isomorphism $\Lambda^*_{\text{const}}(T^n) = H^*(T^n)$.**

Proof: The embedding $\Lambda^*_{\text{const}}(T^n) = \Lambda^*(T^n)^{T_n} \hookrightarrow \Lambda^*(T^n)$ induces an isomorphism on cohomology, however, all constant forms are closed, hence $H^*(\Lambda^*_{\text{const}}(T^n), d) = \Lambda^*_{\text{const}}(T^n)$. ■

Holomorphic vector fields

DEFINITION: Let (M,I) be a complex manifold, and $X \in TM$ a real vector field. It is called **holomorphic** if $\operatorname{Lie}_X(I) = 0$, that is, if the corresponding flow of diffeomorphisms is holomorphic.

CLAIM: Let (M,I) be a complex manifold, and $X \in TM$ a holomorphic vector field. **Then $X^c := I(X)$ is also holomorphic, and commutes with X.**

Proof. Step 1: Assume that X is non-zero at a given point $m \in M$ Solving the appropriate differential equation in holomorphic coordinates, we obtain a coordinate system $z_1, ..., z_n$ in a neighbourhood of m such that $\operatorname{Lie}_X z_i = 0$ for $i > 1$ and $\operatorname{Lie}_{z_1} = 1$. Let x_i, y_i be the corresponding real coordinate system, wih $x_i = \operatorname{Re} z_i$ and $y_i = \operatorname{Im} z_i$. Then $X = \frac{d}{dx_1}$ and $X^c = \frac{d}{dy_1}$.

Step 2: The conditions $\operatorname{Lie}_{X^c}(I) = 0$ and $[X^c, X] = 0$ hold on a closed subset of M, that is, they are true on the closure C of the set of points where $X \neq 0$. Outside of C, the vector field X is identically zero, hence these conditions are also hold. ■

Cartan's formula for Dolbeault differential

LEMMA: Let X be a holomorphic vector field, and $X^c = I(X)$. **Then** $\{d^c, i_X\} = -\operatorname{Lie}_{X^c}$.

Proof: Using $\{IdI^{-1}, i_X\} = I\{d, I^{-1} i_X I\} I^{-1}$, we obtain $\{d^c, i_X\} = -I\{d, i_{X^c}\} I^{-1} = I \operatorname{Lie}_{X^c} I^{-1}$. However, X^c is holomorphic, hence $I \operatorname{Lie}_{X^c} I^{-1} = \operatorname{Lie}_{X^c}$. ■

PROPOSITION: Let X be a holomorphic vector field, and $X^c = I(X)$. **Then** $\{\overline{\partial}, i_X\} = \frac{1}{2}(\operatorname{Lie}_X - \sqrt{-1}\operatorname{Lie}_{X^c})$.

Proof: $\overline{\partial} = \frac{1}{2}(d + \sqrt{-1}\, d^c)$, hence

$$\{\overline{\partial}, i_X\} = \frac{1}{2}\operatorname{Lie}_X + \sqrt{-1}\,\{d^c, i_X\} = \frac{1}{2}(\operatorname{Lie}_X - \sqrt{-1}\operatorname{Lie}_{X^c}).$$

■

REMARK: Let M be a complex manifold equipped with a holomorphic action of the torus T^n. Then the action of T^n commutes with d and d^c. Therefore, the operators d, d^c preserve the eigenspaces of the corresponding Lie algebra. These eigenspaces are components of the weight decomposition. This implies that **the Dolbeault differential $\overline{\partial}$ preserves the weight decomposition.**

Dolbeault cohomology of an elliptic curve

DEFINITION: **An elliptic curve** is a 1-dimensional compact complex manifold $X := \mathbb{C}/\mathbb{Z}^2$.

REMARK: The additive group $\mathbb{C}$ acts on itself by parallel transforms, hence **the 2-dimensional torus T^2 acts on an elliptic curve by holomorphic diffeomorphisms.**

DEFINITION: The T^n-invariant forms on T^n are called **constant**.

DEFINITION: **Dolbeault cohomology** of a complex manifold is $\frac{\ker \overline{\partial}}{\operatorname{im} \overline{\partial}}$.

COROLLARY: **Dolbeault cohomology of an elliptic curve X are represented by the constant forms on X.**

Proof using the Hodge theory: Choose a T^2-invariant Kähler form on X. We have already obtained an isomorphism between de Rham cohomology and the constant forms. Since the constant forms are harmonic, there are no other harmonic forms. Now, $\Delta_{\overline{\partial}} = \frac{1}{2}\Delta_d$, hence

constant forms = $\overline{\partial}$-harmonic forms = Dolbeault cohomology. ∎

In the next slide, **we give a proof which is independent from the Hodge theory.**

Dolbeault cohomology of an elliptic curve (2)

PROPOSITION: Let X be an elliptic curve, and $\Lambda^*(X) = \bigoplus_{\alpha\in\mathbb{Z}^2} \Lambda^*(X)_{p_1,p_2}$ its weight decomposition under the T^2-action. Consider the space T^2-invariant forms $\Lambda^*(X)^{T^2} = \Lambda^*(X)_{0,0}$. **Then the natural embedding $\Lambda^*(X)^{T^2} \hookrightarrow \Lambda^*(X)$ induces an isomorphism of Dolbeault cohomology.**

Proof: Let $\alpha \in \Lambda^*(X)_{p_1,p_2}$ be a $\overline{\partial}$-closed form, with $(p,q) \neq (0,0)$. Suppose, for example, that $p\neq 0$, and X is the generator of the corresponding component of the Lie algebra such that $\operatorname{Lie}_X \alpha = p\sqrt{-1}\,\alpha$. Since X^c belongs to the same Lie algebra, we have $\operatorname{Lie}_{X^c}(\alpha) = v\alpha$, where $v\in\sqrt{-1}\,\mathbb{R}$. Then

$$\frac{\sqrt{-1}\,p+v}{2}\alpha = \frac{1}{2}(\operatorname{Lie}_X - \sqrt{-1}\operatorname{Lie}_{X^c})\alpha = \{\overline{\partial}, i_X\}\alpha = \overline{\partial} i_X\alpha, \quad (***)$$

hence α is $\overline{\partial}$-exact. This implies that $\overline{\partial}$ **has no cohomology on**

$$\bigoplus_{p_1,p_2\neq(0,0)} \Lambda^*(X)_{p_1,p_2}.$$

■

$\overline{\partial}$-exact top forms on an elliptic curve

CLAIM: Let $\eta \in \Lambda^n(T^n)$ be a top form on a torus, and $\nu = \sum_{\alpha\in\mathbb{Z}^n}\nu_\alpha$ its weight decomposition. **Then $\int_{T^n}\nu = \int_{T^n}\nu_0$, where ν_0 denotes the T^n-invariant component.** Moreover, **whenever $\int_{T^n}\nu = 0$, the component ν_0 also vanishes.**

Proof: Let ν be a top form on a compact manifold, equipped with an action of S^1, and $\nu = \sum \nu_i$ its weight decomposition. **Then $\int_M \nu_i = 0$ for all $i \neq 0$.** Indeed, the S^1-action multiplies ν_i by a non-zero number, but the integral is invariant under the action of diffeomorphisms. ■

PROPOSITION: Let $\eta \in \Lambda^2(X)$ be a form on an elliptic curve such that $\int_X \eta = 0$. **Then η is $\overline{\partial}$-exact.**

Proof: Consider the weight decomposition $\eta = \sum_{\alpha\in\mathbb{Z}^2}\eta_\alpha$. Since $\int_M \eta = 0$, the (0,0)-component vanishes, and by (***) the form η is $\overline{\partial}$-exact. ■

Dolbeault cohomology of a disk

COROLLARY: Let $K \subset \mathbb{C}$ be a compact subset, K^0 its interior, and $\eta \in \Lambda^2(K^0)$ a top form smoothly extending to a neighbourhood of K. **Then η is $\overline{\partial}$-exact.**

Proof: Choosing an appropriate lattice $\mathbb{Z}^2 \subset \mathbb{C}$, we may assume that K is a subset of an elliptic curve X. Since η extends to a neighbourhood of K, we can use partition of unity to extend it to a smooth form $\tilde{\eta}$ on X. Applying the weight decomposition $\tilde{\eta} = \sum_{\alpha\in\mathbb{Z}^2} \eta_\alpha$, we obtain that the form $\eta - \eta_{0,0}$ is $\overline{\partial}$-exact. However, the constant part $\eta_{0,0} = \text{const} \cdot dz \wedge d\bar{z} = \text{const} \cdot \overline{\partial}(\bar{z}dz)$ is also $\overline{\partial}$-exact. ∎

COROLLARY: Let $K \subset \mathbb{C}$ be a compact subset, K^0 its interior, and $\mu \in \Lambda^2(K^0)$ a (0,1)-form smoothly extending to a neighbourhood of K. **Then μ is $\overline{\partial}$-exact.**

Proof: By the previous corollary, $\mu \wedge dz$ is $\overline{\partial}$-exact: there exists a (1,0)-form φ such that $\overline{\partial}\varphi = \mu \wedge dz$. However, for any (1,0)-form φ there exists a function ψ such that $\psi dz = \varphi$, which gives $\overline{\partial}\psi = \mu$ because the map $\Lambda^{0,1}(X) \xrightarrow{\wedge dz} \Lambda^{1,1}(X)$ is an isomorphism which commutes with $\overline{\partial}$. ∎

Poincaré-Dolbeault-Grothendieck lemma

DEFINITION: Polydisc D^n is a product of n discs $D \subset \mathbb{C}$.

THEOREM: (Poincaré-Dolbeault-Grothendieck lemma)
Let $\eta \in \Lambda^{p,q}(D^n)$, $q > 0$, be a $\bar\partial$-closed form on a polydisc, smoothly extended to a neighbourhood of its closure $\overline{D^n} \subset \mathbb{C}^n$. **Then η is $\bar\partial$-exact.**

We proved it for $n = 1$. Nextr lecture we prove it for all n.