Complex geometry

lecture 15: Poincaré-Dolbeault-Grothendieck lemma

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Weight decomposition for T^n-action (reminder)

EXERCISE: Consider the n-dimensional torus T^n as a Lie group, $T^n = U(1)^n$. **Prove that any finite-dimensional Hermitian representation of T^n is a direct sum of 1-dimensional representations,** with action of T^n given by $\rho(t_1, ..., t_n)(x) = \exp(2\pi\sqrt{-1}\ \sum_{i=1}^n p_i t_i)x$, for some $p_1, ..., p_n \in \mathbb{Z}^n$, called **the weights** of the 1-dimensional representation.

DEFINITION: Let V be a Hermitian space (possibly infinitely-dimensional) equipped with an action of T^n, and $V_\alpha \subset V$ weight α representations, $\alpha \in \mathbb{Z}^n$. The direct sum $\bigoplus_{\alpha\in\mathbb{Z}^n} V_\alpha$ is called **the weight decomposition** for V if it is dense in V.

THEOREM: Let W be a Hermitian vector space. Then **the Fourier series provide the weight decomposition on $L^2(T^n, W)$.** ■

THEOREM: Let W be a Hermitian representation of T^n. **Then W admits a weight decomposition $V = \widehat{\bigoplus_{\alpha\in\mathbb{Z}^n} W_\alpha}$.**

Proof: We realize W as a subrepresentation in $L^2(T^n, W)$, and use the Fourier series to obtain the weight decomposition of $L^2(T^n, W)$. ■

Weight decomposition for T^n-action on differential forms (reminder)

REMARK: Let M be a manifold with the T^n-action, and

$$\Lambda^*(M) = \widehat{\bigoplus}_{\alpha\in\mathbb{Z}^n}\Lambda^*(M)_{p_1,\dots,p_k}$$

be the weight decomposition on the differential forms. Then the **de Rham differential preserves each term $\Lambda^*(M)_{p_1,\dots,p_k}$.** Indeed, d **commutes with the action of the Lie algebra of T^n, and $\Lambda^*(M)_{p_1,\dots,p_k}$ are its eigenspaces.**

REMARK: Let $\alpha = \sum \alpha_{p_1,\dots,p_k}$ be the weight decomposition. **The forms $\alpha_{p_1,\dots,p_k}$ are obtained by averaging**

$$e^{2\pi\sqrt{-1}\sum_{i=1}^n p_i t_i}\alpha = \mathrm{Av}_{T^n} e^{2\pi\sqrt{-1}\sum_{i=1}^n -p_i t_i}\alpha$$

hence they are smooth.

THEOREM: Let M be a smooth manifold, and T^n a torus acting on M by diffeomorphisms. Denote by $\Lambda^*(M)^{T^n}$ the complex of T^n-invariant differential forms. **Then the natural embedding $\Lambda^*(M)^{T^n} \hookrightarrow \Lambda^*(M)$ induces an isomorphism on de Rham cohomology.**

Constant forms on a torus (reminder)

DEFINITION: Let $T^n = (S^1)^n$ be a compact torus equipped with a action on itself by shifts, and $\Lambda^*_{\text{const}}(M)$. the space of T^n-invariant forms on T^n. These forms are called **constant differential forms**. Clearly, **constant forms have constant coefficients in the usual (flat) coordinates on the torus.**

THEOREM: The natural embedding $\Lambda^*_{\text{const}}(T^n) \hookrightarrow \Lambda^*(T^n)$ **induces an isomorphism $\Lambda^*_{\text{const}}(T^n) = H^*(T^n)$.**

Proof: The embedding $\Lambda^*_{\text{const}}(T^n) = \Lambda^*(T^n)^{T_n} \hookrightarrow \Lambda^*(T^n)$ induces an isomorphism on cohomology, however, all constant forms are closed, hence $H^*(\Lambda^*_{\text{const}}(T^n), d) = \Lambda^*_{\text{const}}(T^n)$. ∎

Holomorphic vector fields (reminder)

DEFINITION: Let (M, I) be a complex manifold, and $X \in TM$ a real vector field. It is called **holomorphic** if $\operatorname{Lie}_X(I) = 0$, that is, if the corresponding flow of diffeomorphisms is holomorphic.

CLAIM: Let (M, I) be a complex manifold, and $X \in TM$ a holomorphic vector field. **Then $X^c := I(X)$ is also holomorphic, and commutes with X.**

LEMMA: Let X be a holomorphic vector field, and $X^c = I(X)$. **Then $\{d^c, i_X\} = -\operatorname{Lie}_{X^c}$.**

Proof: Using $\{IdI^{-1}, i_X\} = I\{d, I^{-1} i_X I\} I^{-1}$, we obtain $\{d^c, i_X\} = -I\{d, i_{X^c}\} I^{-1} = I \operatorname{Lie}_{X^c} I^{-1}$. However, X^c is holomorphic, hence $I \operatorname{Lie}_{X^c} I^{-1} = \operatorname{Lie}_{X^c}$. ■

PROPOSITION: Let X be a holomorphic vector field, and $X^c = I(X)$. **Then $\{\bar\partial, i_X\} = \frac{1}{2}(\operatorname{Lie}_X - \sqrt{-1} \operatorname{Lie}_{X^c})$.**

Proof: $\bar\partial = \frac{1}{2}(d + \sqrt{-1}\, d^c)$, hence

$$\{\bar\partial, i_X\} = \frac{1}{2} \operatorname{Lie}_X + \sqrt{-1}\,\{d^c, i_X\} = \frac{1}{2}(\operatorname{Lie}_X - \sqrt{-1} \operatorname{Lie}_{X^c}).$$

■

Dolbeault cohomology of an elliptic curve (reminder)

PROPOSITION: Let $X = \mathbb{C}/\mathbb{Z}^2$ be an elliptic curve, and $\Lambda^*(X) = \bigoplus_{\alpha\in\mathbb{Z}^2} \Lambda^*(X)_{p_1,p_2}$ its weight decomposition under the T^2-action. Consider the space T^2-invariant forms $\Lambda^*(X)^{T^2} = \Lambda^*(X)_{0,0}$. **Then the natural embedding $\Lambda^*(X)^{T^2} \hookrightarrow \Lambda^*(X)$ induces an isomorphism of Dolbeault cohomology.**

Proof: Let $\alpha \in \Lambda^*(X)_{p_1,p_2}$ be a $\overline{\partial}$-closed form, with $(p,q) \neq (0,0)$. Suppose, for example, that $p \neq 0$, and X is the generator of the corresponding component of the Lie algebra such that $\operatorname{Lie}_X \alpha = p\sqrt{-1}\,\alpha$. Since X^c belongs to the same Lie algebra, we have $\operatorname{Lie}_{X^c}(\alpha) = v\alpha$, where $v \in \sqrt{-1}\,\mathbb{R}$. Then

$$\frac{\sqrt{-1}\,p + v}{2}\alpha = \frac{1}{2}(\operatorname{Lie}_X - \sqrt{-1}\,\operatorname{Lie}_{X^c})\alpha = \{\overline{\partial}, i_X\}\alpha = \overline{\partial} i_X\alpha, \quad (***)$$

hence α is $\overline{\partial}$-exact. This implies that $\overline{\partial}$ **has no cohomology on**

$$\bigoplus_{p_1,p_2\neq(0,0)} \Lambda^*(X)_{p_1,p_2}.$$

■

Dolbeault cohomology of a disk

COROLLARY: Let $K \subset \mathbb{C}$ be a compact subset, K^0 its interior, and $\eta \in \Lambda^{0,1}(K^0)$ a form smoothly extending to a neighbourhood of K. **Then η is $\overline{\partial}$-exact.**

Proof: Choosing an appropriate lattice $\mathbb{Z}^2 \subset \mathbb{C}$, we may assume that K is a subset of an elliptic curve X. Since η extends to a neighbourhood of K, we can use partition of unity to extend it to a smooth form $\tilde{\eta}$ on X. Applying the weight decomposition $\tilde{\eta} = \sum_{\alpha \in \mathbb{Z}^2} \eta_\alpha$, we obtain that the form $\eta - \eta_{0,0}$ is $\overline{\partial}$-exact. However, the constant part $\eta_{0,0} = \text{const} \cdot dz \wedge d\overline{z} = \text{const} \cdot \overline{\partial}(\overline{z} dz)$ (for (1,1)-form) or $\eta_{0,0} = \text{const} \cdot d\overline{z} = \text{const} \cdot \overline{\partial}(\overline{z})$ for (0,1)-form is also $\overline{\partial}$-exact. ■

Poincaré-Dolbeault-Grothendieck lemma

DEFINITION: Polydisc D^n is a product of n discs $D \subset \mathbb{C}$.

THEOREM: (Poincaré-Dolbeault-Grothendieck lemma)
Let $\eta \in \Lambda^{p,q}(D^n)$, $q > 0$, be a $\overline{\partial}$-closed form on a polydisc, smoothly extended to a neighbourhood of its closure $\overline{D^n} \subset \mathbb{C}^n$. **Then η is $\overline{\partial}$-exact.**

We proved it for $n = 1$. Now we prove it for all n.

$\overline{\partial}$-homotopy operator on T^2

From now on, **1-dimensional complex torus is always $\mathbb{C}/\mathbb{Z}[\sqrt{-1}\,]$ and the n-dimensional complex torus T^{2n} is a product of n copies of $T^2 = \mathbb{C}/\mathbb{Z}[\sqrt{-1}\,]$.**

CLAIM: Let $\mu \in \Lambda^{p,q}(M)_{a,b}$ be a form of weight (a,b) on a torus $T^2 = \mathbb{C}/\mathbb{Z}[\sqrt{-1}\,]$, and X the coordinate vector field along the real axis. **Then $\{\overline{\partial}, i_X\}(\mu) = \frac{1}{2}(b + \sqrt{-1}\,a)$.**

Proof: $\{\overline{\partial}, i_X\} = \frac{1}{2}(\operatorname{Lie}_X - \sqrt{-1}\,\operatorname{Lie}_{X^c})$, and X^c is the coordinate vector field along the imaginary axis, acting on μ by multiplication by $\sqrt{-1}\,b$. ■

DEFINITION: Given $\mu = \sum_{a,b\in\mathbb{Z}^2} \mu_{a,b}$ define

$$P(\mu) := \sum_{(a,b)\neq(0,0)} 2(b + \sqrt{-1}\,a)^{-1}\mu_{a,b}.$$

The operator P **commutes with all operators which commute with the T^2-action on itself:** with d, d^c, i_X, i_{X^c}, etc.

COROLLARY: Then $\{\overline{\partial}, Pi_X\}) = \mu - \mu_{0,0}$. In particular, **if μ is $\overline{\partial}$-closed, we also have $\overline{\partial}P(i_X(\mu)) = \mu - \mu_0$.** ■

Homotopy operator γ_k on T^{2n}

Let $U \subset T^{2n}$ be a polydisk. Since U is contractible, all constant (p,q)-forms on a torus with $q>0$ are $\bar\partial$-exact on U: $\bar\partial \bar z_i = \bar\partial(\bar z_i)$, which can be well defined on U because it is contractible.

For any disk $U \subset T^2$, fix a cutoff function ρ_ε which is 1 on U and 0 outside of a contractible ε-neighbourhood of $\overline{U}$. Consider the map $Q: \Lambda^{p,1}(T^2) \longrightarrow \Lambda^{p,0}(T^2)$ taking μ to $\mu_{0,0}$ and replacing any constant summand of form $\alpha\wedge\bar\partial\bar z_i$ by $\rho_\varepsilon \bar z_i \alpha$.

CLAIM: In these assumptions, **we have $\{\bar\partial, \gamma\}(\mu) = \mu$ on U for any form $\mu \in \Lambda^{p,1}(T^2)$,** where $\gamma(\alpha) = P(i_X(\alpha)) + Q(\mu)$.

Proof: If $\mu_{0,0}=0$, we have $Q(\mu)=0$, and this expression becomes $\{\bar\partial, P(i_X)\}) = \mu - \mu_{0,0}$ proven above. If $\mu=\mu_{0,0}$, it becomes $\bar\partial(Q(\mu))|_U = \mu$. ■

Corollary 1: Let $U \subset T^{2n}$ be a polydisk, and ρ_ε a cutoff function which is 1 on U and 0 outside of a contractible ε-neighbourhood of $\overline{U}$. We chose ρ_ε in such a way that $\operatorname{Lie}_{d/dx_i}(\rho_\varepsilon)=0$ at any point $(x_1, ..., x_n)$ such that $|x_i|<1$. Let γ_k denote the operator γ along the k-th component in $T^{2n} = (T^2)^n$, and $\bar\partial_k$ the $\bar\partial$ along this component. **Then $\{\bar\partial_k, \gamma_k\}(\mu) = \mu$ on U for any form μ divisible by $d\bar z_k$, and $\{\bar\partial_k, \gamma_l\}|_U = 0$ for $l \neq k$.** ■

Poincaré-Dolbeault-Grothendieck lemma

THEOREM: (Poincaré-Dolbeault-Grothendieck lemma)
Let $\eta \in \Lambda^{0,p}(D^n)$ be a $\overline{\partial}$-closed form on a polydisc, smoothly extended to a neighbourhood of its closure $\overline{D^n} \subset \mathbb{C}^n$. **Then η is $\overline{\partial}$-exact.**

We prove the following version of Poincaré-Dolbeault-Grothendieck.

THEOREM: Let $U \subset T^{2n}$ be a sufficiently small polydisk, and $\mu \in \Lambda^{p,q}(T^{2n})$ a form with $q > 0$ which is $\overline{\partial}$-closed on U. **Then there exists $\alpha \in \Lambda^{p,q-1}(T^{2n})$ such that $\overline{\partial}\alpha = \mu$ on U.**

Proof. Step 1: Let $\overline{\partial}_i : \Lambda^{p,q}(T^n) \longrightarrow \Lambda^{p,q+1}(T^n)$ be the operator $\alpha \longrightarrow d\overline{z}_i \wedge \frac{d}{d\overline{z}_i}\alpha$, where z_i is i-th coordinate on T^n. **Then $\overline{\partial} = \sum_i \overline{\partial}_i$.** Denote by γ_i the homotopy operator defined above. If $\alpha = d\overline{z}_i \wedge \beta$, one has $\{\overline{\partial}_i, \gamma_i\}(\alpha) = \alpha$. If α contains no monomials divisible by $d\overline{z}_i$, one has

$$\overline{\partial}_i\{\overline{\partial}_i, \gamma_i\}(\alpha) = \overline{\partial}_i\gamma_i\overline{\partial}_i(\alpha) = \{\overline{\partial}_i, \gamma_i\}\overline{\partial}_i\alpha = \overline{\partial}_i\alpha,$$

hence $\overline{\partial}_i(\alpha - \{\overline{\partial}_i, \gamma_i\})|_U = 0$. This implies that $\operatorname{im}\left[\{\overline{\partial}_i, \gamma_i\} - \operatorname{Id}\right]\Big|_U$ **lies in the space $R_i(U)$ of forms without $d\overline{z}_i$ in monomial decomposition and with all coefficients holomorphic as functions on z_i.**

Poincaré-Dolbeault-Grothendieck lemma (2)

THEOREM: Let $U \subset T^{2n}$ be a sufficiently small polydisk, and $\mu \in \Lambda^{p,q}(T^{2n})$ a form with $q > 0$ which is $\overline{\partial}$-closed on U. **Then there exists $\alpha \in \Lambda^{p,q-1}(T^{2n})$ such that $\overline{\partial}\alpha = \mu$ on U.**

Proof. Step 1: Let $\overline{\partial}_i : \Lambda^{p,q}(D^n) \longrightarrow \Lambda^{p,q+1}(D^n)$ be the operator $\alpha \longrightarrow d\overline{z}_i \wedge \frac{d}{d\overline{z}_i}\alpha$, and γ_i the homotopy defined above. Then $\operatorname{im}\left[\{\overline{\partial}_i, \gamma_i\} - \operatorname{Id}\right]\big|_U$ lies in the space $R_i(U)$ of forms without $d\overline{z}_i$ in monomial decomposition and with all coefficients holomorphic as functions on z_i.

Step 2: Let R_i denote the space of forms α on T^{2n} such that $\alpha|_U$ belongs to the space $R_i(U)$ defined above. Properties of γ_i:
(1). $\operatorname{im}\left[\{\overline{\partial}_i, \gamma_i\} - \operatorname{Id}\right] \subset R_i$. **(2).** $\{\overline{\partial}_i, \gamma_j\}|_U = 0$, **if** $i \neq j$. **(3). the restriction** $\left[\{\overline{\partial}_i, \gamma_i\}\right]\Big|_{R_i}$ **vanishes on** U. **(4).** $\gamma_i(R_j) \subset R_j$, $\overline{\partial}_i(R_j) \subset R_j$ **for all** $i \neq j$.
Property (1) is proven in Step 1, property (2) and (4) follow because γ_i is independent from the z_j-coordinate for all $j \neq i$. Finally, (3) follows because for all forms α without $d\overline{z}_i$ in its monomial decomposition one has $\{\gamma_i, \overline{\partial}\}(\alpha) = \gamma_i(\overline{\partial}_i(\alpha))$.

Step 3: Properties (1), (3) and (4) give $\left[\{\overline{\partial}_i, \gamma_i\} - \operatorname{Id}\right](R_{i_1} \cap R_{i_2} \cap ... \cap R_{i_k}) \subset R_i \cap R_{i_1} \cap R_{i_2} \cap ... \cap R_{i_k}$ for $i \notin \{i_1, i_2, ..., i_k\}$, and $\{\overline{\partial}_i, \gamma_i\}\Big|_{R_{i_1} \cap R_{i_2} \cap ... \cap R_{i_k}} = 0$ otherwise.

Poincaré-Dolbeault-Grothendieck lemma (3)

Step 3: Properties (1), (3) and (4) give $\left[\{\overline{\partial}_i, \gamma_i\} - \operatorname{Id}\right](R_{i_1} \cap R_{i_2} \cap ... \cap R_{i_k}) \subset R_i \cap R_{i_1} \cap R_{i_2} \cap ... \cap R_{i_k}$ for $i \notin \{i_1, i_2, ..., i_k\}$, and $\{\overline{\partial}_i, \gamma_i\}\Big|_{R_{i_1} \cap R_{i_2} \cap ... \cap R_{i_k}} = 0$ otherwise.

Step 4: Let $\gamma := \sum_i \gamma_i$. Since $\{\overline{\partial}_i, \gamma_j\} = 0$ for $i \neq j$, Step 3 gives

$$\left[\{\overline{\partial}, \gamma\} - (n-k)\operatorname{Id}\right](R_{i_1} \cap R_{i_2} \cap ... \cap R_{i_k}) \subset \sum_{i \neq i_1, i_2, ..., i_k} R_i \cap R_{i_1} \cap R_{i_2} \cap ... \cap R_{i_k}$$

Step 5: Let $W_0 = \Lambda^*(T^{2n})$, and $W_k \subset W_{k-1}$ the subspace generated by all $R_{i_1} \cap R_{i_2} \cap ... \cap R_{i_k}$ for $i_1 < i_2 < ... < i_k$. **Step 4 implies $\left[\{\overline{\partial}, \gamma\} - (n-k)\operatorname{Id}\right]\Big|_{W_k} \subset W_{k+1}$.**

Step 6: W_n is the space of $(p,0)$-forms holomorphic on U, and it does not contain any (p,q)-forms for $q > 0$. Using induction in $d = n-k$, **we can assume that any $\overline{\partial}$-closed (p,q)-form in W_{k+1} is $\overline{\partial}$-exact when $q > 0$. To prove PDG-lemma, it would suffice to prove the same for any $\overline{\partial}$-closed form $\alpha \in W_k$.** Step 5 gives $(n-k)\alpha - \{\overline{\partial}, \gamma\}(\alpha) = (n-k)\alpha - \overline{\partial}\gamma(\alpha) \in W_{k+1}$, and this form is $\overline{\partial}$-exact by the induction assumption. **This gives $(n-k)\alpha - \overline{\partial}\gamma(\alpha) = \overline{\partial}\eta$**, hence α is $\overline{\partial}$-exact. ■

Hartogs theorem

THEOREM: Let f be a holomorphic function on $\mathbb{C}^n \backslash K$, where $K \subset \mathbb{C}^n$ is a compact, and $n > 1$. **Then f can be extended to a holomorphic function on $\mathbb{C}^n$.**

Proof. Step 1: Replacing K by a bigger compact, we can assume that f is smoothly extended to a small neighbourhood of the closure $\overline{M\backslash K}$. Then f can be extended to a smooth function on $\mathbb{C}^n$, holomorphic outside of K. **Then $\alpha := \overline{\partial}\tilde{f}$ is a $\overline{\partial}$-closed $(0,1)$-form with compact support.**

Step 2: Using the standard open embedding of $\mathbb{C}^n$ to $\mathbb{C}P^n$, we may consider α as a $\overline{\partial}$-closed $(0,1)$-form on $\mathbb{C}P^n$. Since $H^1(\mathbb{C}P^n) = 0$, this gives $\alpha = \overline{\partial}\varphi$, where φ is a continuous function on $\mathbb{C}P^n$. In particular, **φ is bounded on $\mathbb{C}^n \subset \mathbb{C}P^n$.**

Step 3: Since $\overline{\partial}\varphi$ vanishes outside of K, the function φ is holomorphic outside of K. Since bounded holomorphic functions on $\mathbb{C}$ are constant, **φ is constant on any affine line not intersecting K.**

Step 4: This implies that $\varphi = \text{const}$ on the union of all affine lines not intersecting K. Since $n > 1$, the complement of this set is compact. Substracting constant if necessary, we obtain that **φ is a function with compact support**.

Step 5: $\overline{\partial}(\tilde{f} - \varphi) = \alpha - \alpha = 0$, **hence $\tilde{f} - \varphi$ is holomorphic**. However, φ has compact support, and therefore $f = \tilde{f} - \varphi$ outside of a compact. ■

Algebra of supersymmetry of a Kähler manifold: reminder

Let (M, I, g) be a Kaehler manifold, ω its Kaehler form. **On $\Lambda^*(M)$, the following operators are defined.**

0. d, d^*, Δ, because it is Riemannian.

1. $L(\alpha) := \omega \wedge \alpha$

2. $\Lambda(\alpha) := *L * \alpha$. It is easily seen that $\Lambda = L^*$.

3. The Weil operator $W\Big|_{\Lambda^{p,q}(M)} = \sqrt{-1}\,(p-q)$

THEOREM: These operators generate a Lie superalgebra $\mathfrak{a}$ of dimension $(5|4)$**,** acting on $\Lambda^*(M)$. Moreover, the Laplacian Δ is central in $\mathfrak{a}$, hence **$\mathfrak{a}$ also acts on the cohomology of M.**

The odd part of this algebra generates "odd Heisenberg algebra" $\langle d, d^c, d^*, (d^c)^*, \Delta\rangle$, with the only non-zero anticommutator $\{d, d^*\} = \{d^c, (d^c)^*\} = \Delta$.

The even part of this algebra contains an $\mathfrak{sl}(2)$-triple $\langle L, \Lambda, H\rangle$ acting on $\mathfrak{a}^{\text{odd}}$ as on a direct sum of two weight 1 representations ("Kodaira relations"). The Weil element commutes with $\langle L, \Lambda, H, \Delta\rangle$ and acts on $\mathfrak{a}^{\text{odd}}$ via $[W, d] = d^c$, $[W, d^*] = (d^c)^*$.

Inverting $\overline{\partial}$ using the Hodge theory

CLAIM: Let β be a $\overline{\partial}$-exact form, and $\gamma := \Delta^{-1}\overline{\partial}^*\beta$. **Then $\overline{\partial}(\gamma) = \beta$.**

Proof: Indeed,

$$\overline{\partial}^*\beta = \{\overline{\partial}, \overline{\partial}^*\}(\Delta^{-1}\overline{\partial}^*\beta) = \overline{\partial}^*\overline{\partial}\gamma$$

because $(\overline{\partial}^*)^2 = 0$ and Δ^{-1} commutes with $\overline{\partial}^*$. However, $\ker \overline{\partial}^*$ is orthogonal to $\operatorname{im} \overline{\partial}$, hence $\overline{\partial}^*\Big|_{\operatorname{im} \overline{\partial}}$ is injective. Then $\overline{\partial}^*\beta = \overline{\partial}^*\overline{\partial}\gamma$ implies $\beta = \overline{\partial}\gamma$. ■

REMARK: Similarly, **for any d-exact form β, one has $\beta = \Delta^{-1}d^*\beta$.**

dd^c-lemma

THEOREM: Let η be a form on a compact Kähler manifold, satisfying one of the following conditions.
(1). η is an exact (p,q)-form. (2). η is d-exact, d^c-closed.
(3). η is ∂-exact, $\overline{\partial}$-closed.
Then $\eta \in \operatorname{im} dd^c = \operatorname{im} \partial\overline{\partial}$.

Proof: Notice immediately that in all three cases η **is closed and orthogonal to the kernel of** Δ, hence its cohomology class vanishes.

Since η is exact, it lies in the image of Δ. Operator $G_\Delta := \Delta^{-1}$ is defined on $\operatorname{im} \Delta = \ker \Delta^\perp$ and commutes with d, d^c.

In case (1), η is d-exact, and $I(\eta) = \overline{\eta}$ is d-closed, hence η is d-exact, d^c-closed like in (2).

Then $\eta = d\alpha$, where $\alpha := G_\Delta d^* \eta$. Since G_Δ and d^* commute with d^c, the form α is d^c-closed; since it belongs to $\operatorname{im} \Delta = \operatorname{im} G_\Delta$, it is d^c-exact, $\alpha = d^c\beta$ which gives $\eta = dd^c\beta$.

In case (3), we have $\eta = \partial\alpha$, where $\alpha := G_\Delta \partial^* \eta$. Since G_Δ and ∂^* commute with $\overline{\partial}$, the form α is $\overline{\partial}$-closed; since it belongs to $\operatorname{im} \Delta$, it is $\overline{\partial}$-exact, $\alpha = \overline{\partial}\beta$ which gives $\eta = \partial\overline{\partial}\beta$. ■

Massey products

Let $a, b, c \in \Lambda^*(M)$ be closed forms on a manifold M with cohomology classes $[a], [b], [c]$ satisfying $[a][b] = [b][c] = 0$, and $\alpha, \gamma \in \Lambda^*(M)$ forms which satisfy $d(\alpha) = a \wedge b$, $d(\gamma) = b \wedge c$. Denote by $L_{[a]}, L_{[c]} : H^*(M) \longrightarrow H^*(M)$ the operation of multiplication by the cohomology classes $[a], [c]$.

Then $\alpha \wedge c - a \wedge \gamma$ is a closed form, and its cohomology class is well-defined modulo $\operatorname{im} L_{[a]} + \operatorname{im} L_{[c]}$.

DEFINITION: Cohomology class $\alpha \wedge c - a \wedge \gamma$ is called **Massey product of a, b, c.**

PROPOSITION: On a Kähler manifold, Massey products vanish.

Proof: Let a, b, c be harmonic forms of pure Hodge type, that is, of type (p, q) for some p, q. Then ab and bc are exact pure forms, hence $ab, bc \in \operatorname{im} dd^c$ by dd^c-lemma. This implies that $\alpha := d^* G_\Delta(ab)$ and $\gamma := d^* G_\Delta(bc)$ are d^c-exact. Therefore $\mu := \alpha \wedge c - a \wedge \gamma$ is a d^c-exact, d-closed form. **Applying dd^c-lemma again, we obtain that μ is dd^c-exact, hence its cohomology class vanish.**

■