Complex geometry

lecture 16: Riemann-Hilbert correspondence

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Sheaves

DEFINITION: An open cover of a topological space X is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$.

REMARK: The definition of a sheaf below **is a more abstract version of the notion of "sheaf of functions"** defined previously.

DEFINITION: A **presheaf** on a topological space M is a collection of vector spaces (or abelian groups) $\mathcal{F}(U)$, for each open subset $U \subset M$, together with **restriction maps** $R_{UW}\mathcal{F}(U) \longrightarrow \mathcal{F}(W)$ defined for each $W \subset U$, such that for any three open sets $W \subset V \subset U$, $R_{UW} = R_{UV} \circ R_{VW}$. Elements of $\mathcal{F}(U)$ are called **sections of $\mathcal{F}$ over U**, and the restriction map often denoted $f|_W$

DEFINITION: A presheaf $\mathcal{F}$ is called **a sheaf** if for any open set U and any cover $U = \bigcup U_I$ the following two conditions are satisfied.

1. Let $f \in \mathcal{F}(U)$ be a section of $\mathcal{F}$ on U such that its restriction to each U_i vanishes. **Then $f = 0$.**

2. Let $f_i \in \mathcal{F}(U_i)$ be a family of sections compatible on the pairwise intersections: $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for every pair of members of the cover. **Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.**

Morphisms of sheaves

DEFINITION: Let $\mathcal{B}, \mathcal{B}'$ be sheaves on M. **A sheaf morphism** from $\mathcal{B}$ to $\mathcal{B}'$ is a collection of homomorphisms $\mathcal{B}(U) \longrightarrow \mathcal{B}'(U)$, defined for each open subset $U \subset M$, and compatible with the restriction maps:

$$\begin{array}{ccc} \mathcal{B}(U) & \longrightarrow & \mathcal{B}'(U) \\ \downarrow & & \downarrow \\ \mathcal{B}(U_1) & \longrightarrow & \mathcal{B}'(U_1) \end{array}$$

DEFINITION: **A sheaf isomorphism** is a homomorphism $\Psi: \mathcal{F}_1 \longrightarrow \mathcal{F}_2$, for which there exists an homomorphism $\Phi: \mathcal{F}_2 \longrightarrow \mathcal{F}_1$, such that $\Phi \circ \Psi = \mathrm{Id}$ and $\Psi \circ \Phi = \mathrm{Id}$.

Sheaves of modules

REMARK: Let $A : \varphi \longrightarrow B$ be a ring homomorphism, and V a B-module. **Then V is equipped with a natural A-module structure: $av := \varphi(a)v$.**

DEFINITION: Let $\mathcal{F}$ be a sheaf of rings on a topological space M, and $\mathcal{B}$ another sheaf. It is called **a sheaf of $\mathcal{F}$-modules** if for all $U \subset M$ the space of sections $\mathcal{B}(U)$ is equipped with a structure of $\mathcal{F}(U)$-module, and for all $U' \subset U$, the restriction map $\mathcal{B}(U) \xrightarrow{\varphi_{U,U'}} \mathcal{B}(U')$ is a homomorphism of $\mathcal{F}(U)$-modules (use the remark above to obtain a structure of $\mathcal{F}(U)$-module on $\mathcal{B}(U')$).

DEFINITION: A **free sheaf of modules** $\mathcal{F}^n$ over a ring sheaf $\mathcal{F}$ maps an open set U to the space $\mathcal{F}(U)^n$.

DEFINITION: Locally free sheaf of modules over a sheaf of rings $\mathcal{F}$ is a sheaf of modules $\mathcal{B}$ satisfying the following condition. For each $x \in M$ there exists a neighbourhood $U \ni x$ such that the restriction $\mathcal{B}|_U$ is free.

DEFINITION: A vector bundle on a smooth manifold M is a locally free sheaf of $C^\infty M$-modules.

EXAMPLE: Clearly, **tangent bundle is a vector bundle**.

Locally constant sheaves

DEFINITION: Let $\mathcal{F}$ be a sheaf on M which takes a connected non-empty open subset $U \subset M$ to a vector space or abelian group $\mathbb{V}$. Extend $\mathcal{F}$ to all open sets using the gluing axiom. Then $\mathcal{F}$ is called **the constant sheaf**, denoted $\mathbb{V}_M$.

EXERCISE: Prove that **the constant sheaf $\mathbb{V}_M$ exists, and is unique up to isomorphism.**

EXERCISE: Let W be an open set in M, and S_W its set of connected components. Prove that $\mathbb{V}_M(W) = \mathbb{V}^{|S_W|}$.

DEFINITION: A **locally constant sheaf** is a sheaf which is locally isomorphic to a constant sheaf.

EXAMPLE: Let $\pi : M' \longrightarrow M$ be a covering. Given $U \subset M$, let S_U be the set of connected components of $\pi^{-1}(U)$, and set $\mathcal{F}(U) = \mathbb{V}^{|S_W|}$. We are going to define the restriction map r as follows. For an open subset $W \subset U$, consider the map $S_W \longrightarrow S_U$ induced by the natural embedding $\pi^{-1}(W) \stackrel{j}{\hookrightarrow} \pi^{-1}(U)$. For each direct sum component $\mathbb{V}_u \subset \mathbb{V}^{|S_U|}$ corresponding to $u \in \operatorname{im} j$, let $r_u : \mathbb{V}_u \longrightarrow \mathbb{V}_{j(u)}$ be identity. For a component $\mathbb{V}_u \subset \mathbb{V}^{|S_U|}$ corresponding to $u \notin \operatorname{im} j$, we set $r_u = 0$. Then $r := \bigoplus_{u\in S_U} r_u : \bigoplus_{u \in S_U} \mathbb{V} \longrightarrow \bigoplus_{w\in S_W} \mathbb{V}$. **This defines a locally constant sheaf on M (prove it).**

Étalé space of a sheaf

DEFINITION: Let $\mathcal{F}$ be a sheaf on M, and $U, V \supset x$ be two open set containing $x \in M$. Two sections $f \in \mathcal{F}(U)$, $g \in \mathcal{F}(V)$ are called **equivalent in x** if there exists an open set $W \ni x$ such that $W \subset U \cap V$ and $f|_W = g|_W$. **A germ of a sheaf $\mathcal{F}$ in x** is a class of equivalence of sections of $\mathcal{F}$ in all open sets $U \ni x$ under this equivalence relation. **The stalk** of a sheaf $\mathcal{F}$ in x is the space $\mathcal{F}_x$ of all germs in x.

DEFINITION: Let $E(\mathcal{F})$ be the set of all stalks of a sheaf $\mathcal{F}$ in all points $x \in M$. A germ $f \in \mathcal{F}_m$ is called **a limit of a sequence of germs $f_i \in \mathcal{F}_{m_i}$** if $\lim_i m_i = m$ and there exists a section $\tilde{f}$ of $\mathcal{F}$ over $U \ni x$ such that almost all f_i are germs of $\tilde{f}$. The **étalé topology** on $E(\mathcal{F})$ is defined as follows: a subset $K \subset E(\mathcal{F})$ is **closed in étalé topology** if it contains all its limit points.

REMARK: Usually $E(\mathcal{F})$ **is non-Hausdorff**.

Étalé space of a constant sheaf

CLAIM: Let $\mathcal{F} = \mathbb{V}_M$ be a constant sheaf on a manifold, and $x \in M$ a connected subset. **Then the space of germs of $\mathcal{F}$ in x is equal to $\mathbb{V}$.**

Proof: Since $\mathcal{F}$ is constant, the set of its sections on any connected open set is equal to $\mathbb{V}$. This gives a natural map $r_x := \mathcal{F}(U) \longrightarrow \mathbb{V}$: we restrict $f \in \mathcal{F}(U)$ to a connected component U_1 of U containing x, and obtain an element of $\mathbb{V}$. **Clearly, two sections f, g are equivalent in K if and only if $r_x(f) = r_x(g)$.** This identifies $\mathbb{V}$ with the set of equivalence classes of sections in x. ■

Corollary 1: Let $\mathcal{F} = \mathbb{V}_M$ be a constant sheaf on a manifold. **Then the étalé space $E(\mathcal{F})$ of $\mathcal{F}$ is identified with $\mathbb{V}$ disconnected copies of M.**

Proof: Indeed, a sequence $f_i \in \mathcal{F}_{m_i}$ converges to f if $\lim_i m_i = m$ and $r_{m_i}(f_i) = r_m(f)$ for almost all i. ■

Local systems

DEFINITION: **Category of coverings** of M is category $\mathcal{C}$ with $\mathcal{O}b(\mathcal{C})$ all coverings and morphisms continuous maps of coverings compatible with projections to M.

DEFINITION: Let $\pi_1: \ M_1 \longrightarrow M$, $\pi_2: \ M_2 \longrightarrow M$ be continuous maps. **Fibered product** $M_1 \times_M M_2$ is the subset of $M_1 \times M_2$ defined as $M_1 \times_M M_2 := \{(x,y) \in M_1 \times M_2 \ \ | \ \ \pi_1(x) = \pi_2(y)\}$, with induced topology.

EXERCISE: Prove that **a fibered product of coverings is a covering**.

DEFINITION: **An abelian group structure on a covering $\pi_1: \ M_1 \longrightarrow M$** is a morphism of coverings $\mu: \ M_1 \times_M M_1 \longrightarrow M_1$ together with a morphism $e: \ M \longrightarrow M_1$ from a trivial covering to M_1 and $\in \operatorname{Hom}_M(M_1)$ such that μ defines an additive structure of an abelian group on the set $\pi_1^{-1}(x)$ for each $x \in M$, with $e(x)$ a unit in this group and a the inverse.

REMARK: If, in addition, we have a group homomorphism $\mathbb{R}^* \longrightarrow \operatorname{Aut}_M(M_1, M_1)$ which equips each $\pi_1^{-1}(x)$ with a structure of a vector space, we obtain **a structure of a vector space on a covering**.

DEFINITION: **A local system** is a covering with a structure of an abelian group or a vector space.

Étalé space of a locally constant sheaf

THEOREM: Let $\mathcal{F} = \mathbb{V}_M$ be a locally constant sheaf on a manifold. **Then its étalé space $E(\mathcal{F})$ is a covering of M.**

Proof: Immediately follows from Corollary 1. ■

THEOREM: Category of locally constant sheaves is equivalent to the category of local systems.

Proof: Let $\mathcal{F}$ be a locally constant sheaf, and $E(\mathcal{F})$ its etale space. Then $E(\mathcal{F})$ is a covering of M. The structure of vector space on germs defines the structure of vector space on $E(\mathcal{F})$. **This gives a functor from locally constant sheaves to local systems.**

Conversely, let $\pi: M_1 \longrightarrow M$ be a local system, and $\mathcal{F}(U)$ be the space of the sections of $\pi^{-1}(U) \xrightarrow{\pi} U$. Then $\mathcal{F}(U)$ is a vector space. The correspondence $U \longrightarrow \mathcal{F}(U)$ gives a sheaf, which is clearly locally constant. ■

Connections (reminder)

Notation: Let M be a smooth manifold, TM its tangent bundle, $\Lambda^i M$ the bundle of differential i-forms, $C^\infty M$ the smooth functions. **The space of sections of a bundle B is denoted by B.**

DEFINITION: A **connection** on a vector bundle B is an operator $\nabla : B \longrightarrow B \otimes \Lambda^1 M$ satisfying $\nabla(fb) = b \otimes df + f\nabla(b)$, where $f \longrightarrow df$ is de Rham differential. When X is a vector field, we denote by $\nabla_X(b) \in B$ the term $\langle \nabla(b), X\rangle$.

REMARK: When $M = [0, a]$ is an interval, any bunlde B on M is trivial. Let $b_1, ..., b_n$ be a basis in B. Then ∇ can be written as

$$\nabla_{d/dt}\left(\sum f_i b_i\right) = \sum_i \frac{df_i}{dt} b_i + \sum f_i \nabla_{d/dt} b_i$$

with the last term linear on f. Therefore, the equation $\nabla_{d/dt}(b) = 0$ is a first order ODE, and **it has a unique solution for any initial value $b_0 = b\Big|_{\{0\}}$.**

Curvature

Let $\nabla: B \longrightarrow B \otimes \Lambda^1 M$ be a connection on a vector bundle B. **We extend ∇ to an operator**

$$B \xrightarrow{\nabla} \Lambda^1(M) \otimes B \xrightarrow{\nabla} \Lambda^2(M) \otimes B \xrightarrow{\nabla} \Lambda^3(M) \otimes B \xrightarrow{\nabla} ...$$

using the Leibnitz identity $\nabla(\eta \otimes b) = d\eta \otimes b + (-1)^{\tilde{\eta}} \eta \wedge \nabla b$.

REMARK: This operation is well defined, because

$$\nabla(\eta \otimes fb) = d\eta \otimes fb + (-1)^{\tilde{\eta}} \eta \wedge \nabla(fb) =$$
$$d\eta \otimes fb + (-1)^{\tilde{\eta}} \eta \wedge df \otimes b + f\eta \wedge \nabla b = d(f\eta) \otimes b + f\eta \wedge \nabla b = \nabla(f\eta \otimes b)$$

REMARK: Sometimes $\Lambda^2(M) \otimes B \xrightarrow{\nabla} \Lambda^3(M) \otimes B$ is denoted d_∇.

DEFINITION: The operator $\nabla^2: B \longrightarrow B \otimes \Lambda^2(M)$ is called **the curvature** of ∇.

REMARK: The algebra of differential forms with coefficients in $\operatorname{End} B$ **acts on $\Lambda^* M \otimes B$** via $\eta \otimes a(\eta' \otimes b) = \eta \wedge \eta' \otimes a(b)$, where $a \in \operatorname{End}(B)$, $\eta, \eta' \in \Lambda^* M$, and $b \in B$. **This is the formula expressing the action of ∇^2 on $\Lambda^* M \otimes B$.**

Riemann-Hilbert correspondence

DEFINITION: A connection is **flat** if its curvature vanishes.

THEOREM: Let M be a connected manifold, $\mathcal{C}_1$ the category of representations of $\pi_1(M)$, and $\mathcal{C}_2$ the category of local systems. **Then the categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are naturally equivalent.**

Proof: Follows from the equivalence between locally constant sheaves and local systems. ■

THEOREM: The categories $\mathcal{C}_1$ and $\mathcal{C}_2$ **are naturally equivalent to the category of vector bundles on M equipped with flat connection.**

We prove it later in this lecture.

Curvature and commutators

CLAIM: Let $X, Y \in TM$ be vector fields, (B, ∇) a bundle with connection, and $b \in B$ its section. Consider the operator

$$\Theta_B^*(X, Y, b) := \nabla_X \nabla_Y b - \nabla_Y \nabla_X b - \nabla_{[X,Y]} b$$

Then $\Theta_B^*(X, Y, b)$ is linear in all three arguments.

Proof. Step 1: The term $\Theta_B^*(X, Y, fb)$ **has 3 components: one which is C^∞-linear in f, one which takes first derivative and one which takes the second derivative.** The first derivative part is

$\operatorname{Lie}_Y f \nabla_X b + \operatorname{Lie}_X f \nabla_Y b - \operatorname{Lie}_Y f \nabla_X b - \operatorname{Lie}_X f \nabla_Y b - \operatorname{Lie}_{[X,Y]} fb = -\operatorname{Lie}_{[X,Y]} fb,$

the second derivative part is $\operatorname{Lie}_X \operatorname{Lie}_Y (f) b - \operatorname{Lie}_Y \operatorname{Lie}_X (f) b = \operatorname{Lie}_{[X,Y]} f$, they cancel. Therefore, **$\Theta_B^*(X, Y, b)$ is C^∞-linear in b.**

Step 2: Since $[X, fY] = \operatorname{Lie}_X fY + f[X, Y]$, we have $\nabla_{[X,fY]} b = f\nabla_{[X,Y]} b + \operatorname{Lie}_X f \nabla_Y b$.

Step 4: The term $\Theta_B^*(X, fY, b)$ has two components, f-linear and the component with first derivatives in f. Step 2 implies that the component with derivative of first order is $\operatorname{Lie}_X f \nabla_Y b - \operatorname{Lie}_X f \nabla_Y b = 0$. ■

Curvature and commutators (2)

REMARK:

$$\Theta_B^*(X,Y,b) := \nabla_X\nabla_Y b - \nabla_Y\nabla_X b - \nabla_{[X,Y]}b$$

is another definition of the curvature. **The following theorem shows that it is equivalent to the usual definition**.

THEOREM: Consider $\Theta_B^* : TM \otimes TM \otimes B \longrightarrow B$ as a 2-form with coefficients in $\operatorname{End}(B)$. **Then $\Theta_B^* = \Theta_B$,** where $\Theta_B = \nabla^2$ is the usual curvature.

Proof. Step 1: Since $\Theta_B^*(X,Y)$, $\Theta_B(X,Y)$ are linear in X,Y, it would suffice to prove this equality for coordinate vector fields X,Y.

Step 2: Consider the operator $i_X : \Lambda^i M \otimes B \longrightarrow \Lambda^{i-1} M \otimes B$ of convolution with a vector field X. Writing $\nabla = d + A$, where $A \in \Lambda^1 M \otimes \operatorname{End} B$, we obtain $\nabla_X = \operatorname{Lie}_X + A(X)$, which gives $[\nabla_X, i_Y] = [\operatorname{Lie}_X, i_Y] = 0$ when X,Y are coordinate vector fields.

Step 3:

$$\nabla^2(b)(X,Y) = (i_X i_Y - i_X i_Y)\nabla^2(b) = i_Y\nabla_X\nabla b - i_X\nabla_Y\nabla b = \nabla_X\nabla_Y b - \nabla_Y\nabla_X b.$$

■

Parallel transport along the connection (reminder)

REMARK: When $M = [0, a]$ is an interval, any bundle B on M is trivial. Let $b_1, ..., b_n$ be a basis in B. Then ∇ can be written as

$$\nabla_{d/dt}\left(\sum f_i b_i\right) = \sum_i \frac{df_i}{dt} b_i + \sum f_i \nabla_{d/dt} b_i$$

with the last term linear on f.

THEOREM: Let B be a vector bundle with connection over $\mathbb{R}$. Then for each $x \in \mathbb{R}$ and each vector $b_x \in B|_x$ **there exists a unique section $b \in B$ such that $\nabla b = 0$, $b|_x = b_x$.**

Proof: This is existence and uniqueness of solutions of an ODE $\frac{db}{dt} + A(b) = 0$. ■

DEFINITION: Let $\gamma : [0, 1] \longrightarrow M$ be a smooth path in M connecting x and y, and (B, ∇) a vector bundle with connection. Restricting (B, ∇) to $\gamma([0, 1])$, we obtain a bundle with connection on an interval. Solve an equation $\nabla(b) = 0$ for $b \in B\Big|_{\gamma([0,1])}$ and initial condition $b|_x = b_x$. This process is called **parallel transport** along the path via the connection. The vector $b_y := b|_y$ is called **vector obtained by parallel transport of b_x along γ**.

Holonomy group

DEFINITION: (Cartan, 1923) Let (B,∇) be a vector bundle with connection over M. For each loop γ based in $x \in M$, let $V_{\gamma,\nabla}: \ B|_x \longrightarrow B|_x$ be the corresponding parallel transport along the connection. The **holonomy group** of (B,∇) is a group generated by $V_{\gamma,\nabla}$, for all loops γ. If one takes all contractible loops instead, $V_{\gamma,\nabla}$ generates **the local holonomy**, or **the restricted holonomy** group.

REMARK: Let $B_1 = B^{\otimes n} \otimes (B^*)^{\otimes m}$ be a tensor power of B. The connection on B gives the connection on B_1. Since parallel transport is compatible with the tensor product, **the holonomy representation, associated with B_1, is the corresponding tensor power of $B|_x$.**

DEFINITION: Let B be a vector bundle, and Ψ a section of its tensor power. We say that **connection ∇ preserves Ψ** if $\nabla(\Psi) = 0$. In this case we also say that the tensor Ψ is **parallel** with respect to the connection.

Flat bundles

REMARK: $\nabla(\Psi) = 0$ is equivalent to Ψ being a solution of $\nabla(\Psi) = 0$ on each path γ. This means that **parallel transport preserves Ψ.**

We obtained

COROLLARY: A section of the tensor power of B is parallel if and only if it is holonomy invariant.

DEFINITION: A bundle is **flat** if its curvature vanishes.

The following theorem will be proven later today.

THEOREM: Let (B, ∇) be a vector bundle with connection over a simply connected manifold. **Then B is flat if and only if its holonomy group is trivial**.

Fiber of a locally free sheaf

DEFINITION: Recall that **a vector bundle** is a locally free sheaf of modules over $C^\infty M$. A vector bundle is called **trivial** if it is isomorphic to $(C^\infty M)^n$.

DEFINITION: Let $\mathcal{B}$ be an n-dimensional locally free sheaf of C^∞-modules on M, $x \in M$ a point, $\mathfrak{m}_x \subset C^\infty M$ an ideal of $x \in M$ in $C^\infty M$. Define **the fiber** of $\mathcal{B}$ in x as a quotient $\mathcal{B}(M)/\mathfrak{m}\mathcal{B}$. A fiber of $\mathcal{B}$ is denoted $\mathcal{B}|_x$.

REMARK: A fiber of a vector bundle of rank n is an n-dimensional vector space.

REMARK: Let $\mathcal{B} = C^\infty M^n$, and $b \in \mathcal{B}|_x$ a point of a fiber, represented by a germ $\varphi \in \mathcal{B}_x = C_m^\infty M^n$, $\varphi = (f_1, ..., f_n)$. Consider a map Ψ from the set of all fibers $\mathcal{B}$ to $M \times \mathbb{R}^n$, mapping $(x, \varphi = (f_1, ..., f_n))$ to $(f_1(x), ..., f_n(x))$. **Then Ψ is bijective.** Indeed, $\mathcal{B}|_x = \mathbb{R}^n$.

Total space of a vector bundle

DEFINITION: Let $\mathcal{B}$ be an n-dimensional locally free sheaf of C^∞-modules. Denote the set of all vectors in all fibers of $\mathcal{B}$ over all points of M by $\operatorname{Tot}\mathcal{B}$. Let $U \subset M$ be an open subset of M, with $\mathcal{B}|_U$ a trivial bundle. Using the local bijection $\operatorname{Tot}\mathcal{B}(U) = U \times \mathbb{R}^n$ we consider topology on $\operatorname{Tot}\mathcal{B}$ induced by open subsets in $\operatorname{Tot}\mathcal{B}(U) = U \times \mathbb{R}^n$ for all open subsets $U \subset M$ and all trivializations of $\mathcal{B}|_U$. Then $\operatorname{Tot}\mathcal{B}$ is called **a total space of a vector bundle $\mathcal{B}$.**

CLAIM: The space $\operatorname{Tot}\mathcal{B}$ with this topology **is a locally trivial fibration over M, with fiber $\mathbb{R}^n$.**

REMARK: Let B be a vector bundle on M, and $\psi \in B^*$ a section of its dual. Then ψ defines a function $x \longrightarrow \langle \psi, x\rangle$ on its total space $\operatorname{Tot}(B) \xrightarrow{\pi} M$, linear on fibers of π. This gives a **bijective correspondence between sections of B^* and functions on $\operatorname{Tot}(B)$ linear on fibers.**

This gives the following claim

CLAIM: Let B be a vector bundle and $\operatorname{Sym}^* B^*$ the direct sum of all symmetric tensor powers of B^*. **Then the ring of sections of $\operatorname{Sym}^* B^*$ is identified with the ring of all smooth functions on $\operatorname{Tot} B \xrightarrow{\pi} M$ which are polynomial on fibers of π.** ■

Polynomial functions on $\operatorname{Tot}(B)$

CLAIM: Let D be the space of derivations $\delta: \mathbb{R}[x_1, ..., x_n] \longrightarrow C^\infty \mathbb{R}^n$. **Then D is the space of derivations of the ring $C^\infty \mathbb{R}^n$.** ■

EXERCISE: Prove it.

The same argument brings the following

CLAIM 1: Let D be the space of derivations $\delta: \operatorname{Sym}^* B^* \longrightarrow C^\infty(\operatorname{Tot} B)$. **Then D is the space of derivations of the ring $C^\infty(\operatorname{Tot} B)$.**

Proof: Indeed, any derivation which vanishes on fiberwise polynomial functions vanishes everywhere on $C^\infty(\operatorname{Tot} B)$. ■

Vector fields on $\operatorname{Tot}(B)$

THEOREM: Let (B, ∇) be a bundle on M with connection, and $X \in TM$ a vector field. **Then there exists a vector field $\tau_\nabla(X)$ on $\operatorname{Tot}(B)$ mapping a section $u \in \operatorname{Sym}^* B^*$ to $\nabla_X u$.**

Proof: Let $u, v \in \operatorname{Sym}^* B^*$, and $uv \in \operatorname{Sym}^* B^*$ their product. Then $\nabla_x(uv) = u\nabla_x v + v\nabla_x u$ because $\nabla(b_1 \otimes b_2) = \nabla(b_1) \otimes b_2 + b_1 \otimes \nabla(b_2)$. Therefore, $\tau_\nabla(X)(u) := \nabla_x(u)$ is a derivation of the ring of functions on $\operatorname{Tot}(B)$ which are polynomial on fibers. By Claim 1, any such derivation can be uniquely extended to a vector field on $\operatorname{Tot}(B)$. ■

DEFINITION: Let (B, ∇) be a bundle with connection on M. The corresponding **Ehresmann connection** on $\operatorname{Tot}(B)$ is the distribution $E_\nabla \subset T\operatorname{Tot}(B)$ obtained as $\tau_\nabla(TM)$.

Vector fields on $\operatorname{Tot}(B)$ and parallel sections

CLAIM 2: Let (B, ∇) be a bundle with connection, and $\pi : \operatorname{Tot}(B) \longrightarrow M$ the standard projection, and $T_\pi \operatorname{Tot}(B) = \ker D\pi$ is the vertical tangent space (Lecture 14).

(i) **Then $T \operatorname{Tot} B = E_\nabla \oplus T_\pi \operatorname{Tot}(B)$, where E_∇ is the Ehresmann connection.**

(ii) Moreover, **a section f of B is parallel if an only if its image $f(M) \subset \operatorname{Tot}(B)$ is tangent to E_∇.**

Proof: The second assertion is clear from the definition: **a section b is tangent to E_∇ if it is preserved by all vector fields $a = \tau_\nabla(X)$ generating E_∇.** In this case $\operatorname{Lie}_a(\tilde{b}) = 0$, where $\tilde{b}$ is a function on $\operatorname{Tot}(B^*)$ defined by b. However, $\operatorname{Lie}_a(\tilde{b}) = \nabla_X(b)$ where $\nabla_X(b)$ is a function on $\operatorname{Tot}(B^*)$ associated with $\nabla_X(b)$. Therefore, $\operatorname{Lie}_a(\tilde{b}) = 0 \Leftrightarrow \nabla_X(b) = 0$.

To prove (i), we notice that $D\pi\big|_{E_\nabla} : E_\nabla \longrightarrow TM$ is an isomorphism at every point of $\operatorname{Tot} B$. Indeed, these bundles have the same rank, and for each $\tau_\nabla(X) \in E_\nabla$, this vector field acts on functions pulled back from M as Lie_X, hence $D\pi\big|_{E_\nabla}$ is injective. ■

Bundles with trivial holonomy

THEOREM: Let (B, ∇) be a vector bundle with connection over a simply connected manifold. **Then B is flat if and only if its holonomy group is trivial**.

Proof: Let B be a flat bundle on M, and $X, Y \in TM$ commuting vector fields. Then $\nabla_X : B \longrightarrow B$ commutes with ∇_Y. Then the Ehresmann connection bundle $E_\nabla \subset T\,\mathrm{Tot}\,B$ is generated by commuting vector fields $\tau_\nabla(X)$, $\tau_\nabla(Y)$, ..., hence it is involutive. By Frobenius theorem, every point $b \in \mathrm{Tot}(B)$ is contained in a leaf $\mathcal{E}$ of the corresponding foliation, tangent to E_∇. By Claim 2, such a leaf is a parallel section of B. The projection from $\mathcal{E}$ to M is a covering. Since M is simply connected, $\mathcal{E} = \mathcal{M}$, and B is trivialized by parallel sections.

Conversely, assume that B has trivial holonomy. Then $\mathrm{Tot}(B) = M \times B|_x$ because each point is contained in a unique parallel section, hence the bundle E_∇ is involutive. Then $[\nabla_X, \nabla_Y] = 0$ for any commuting $X, Y \in TM$, and the curvature vanishes. ■

Corollary 1: Let B be a flat vector bundle on a simply connected, connected manifold M. **Then for each $x \in M$ and each $b \in B|_x$, there exists a unique parallel section of B passing through b.** ■

Riemann-Hilbert correspondence

THEOREM: The category of locally constant sheaves of vector spaces **is naturally equivalent to the category of vector bundles on M equipped with flat connection.**

Proof. Step 1: Consider a constant sheaf $\mathbb{R}_M$ on M. This is a sheaf of rings, and any locally constant sheaf is a sheaf of $\mathbb{R}_M$-modules.

Let $\mathbb{V}$ be a locally constant sheaf, and $B := \mathbb{V} \otimes_{\mathbb{R}_M} \mathbb{C}^\infty M$. Since $\mathbb{V}$ is locally constant, the sheaf B is a locally free sheaf of C^∞-modules, that is, a vector bundle. Let $U \subset M$ be an open set such that $\mathbb{V}|_U$ is constant. If $v_1, ..., v_n$ is a basis in $\mathbb{V}(U)$, all sections of $B(U)$ have a form $\sum_{i=1}^n f_i v_i$, where $f_i \in C^\infty U$. Define the connection ∇ by $\nabla\left(\sum_{i=1}^n f_i v_i\right) = \sum df_i \otimes v_i$. This connection is flat because $d^2 = 0$. It is independent from the choice of v_i because v_i is defined canonically up to a matrix with constant coefficients. **We have constructed a functor from locally constant sheaves to flat vector bundles**.

Step 2: Let now (B, ∇) be a flat bundle over M. The functor to locally constant sheaves takes $U \subset M$ and maps it to the space of parallel sections of B over U. This defines a sheaf $\mathbb{B}(U)$. For any simply connected U, and any $x \in M$, the space $\mathbb{B}(U)$ is identified with a vector space $B|_x$ (Corollary 1), hence $\mathbb{B}(U)$ is locally constant. Clearly, $B = \mathbb{B} \otimes_{\mathbb{R}_M} \mathbb{C}^\infty M$, hence **this construction gives an inverse functor to $\mathbb{V} \mapsto \mathbb{V} \otimes_{\mathbb{R}_M} \mathbb{C}^\infty M$.** ■