Locally conformally Kähler manifolds

lecture 1

Misha Verbitsky

HSE and IUM, Moscow

February 10, 2014

Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X,Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

REMARK: The commutator defines a $C^\infty M$-linear map $N := \Lambda^2(T^{1,0}) \longrightarrow T^{0,1}M$, called **the Nijenhuis tensor** of I. **One can represent N as a section of $\Lambda^{2,0}(M) \otimes T^{0,1}M$.**

Kähler manifolds

DEFINITION: An Riemannian metric g on an almost complex manifold M is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^2(M)$ is called **the Hermitian form** of (M, I, g).

REMARK: It is $U(1)$-invariant, hence **of Hodge type (1,1)**.

DEFINITION: A complex Hermitian manifold (M, I, ω) is called **Kähler** if $d\omega = 0$. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M, and ω **the Kähler form**.

Definition: Let $M = \mathbb{C}P^n$ be a complex projective space, and g a $U(n+1)$-invariant Riemannian form. It is called **Fubini-Study form on $\mathbb{C}P^n$**. The Fubini-Study form is obtained by taking arbitrary Riemannian form and averaging with $U(n+1)$ using the Haar measure on $U(n+1)$.

EXERCISE: Prove that **the Fubini-Study form is unique** (up to a constant multiplier).

Examples of Kähler manifolds.

Remark: For any $x \in \mathbb{C}P^n$, the stabilizer $St(x)$ is isomorphic to $U(n)$. Fubini-Study form on $T_x\mathbb{C}P^n = \mathbb{C}^n$ is $U(n)$-invariant, hence unique up to a constant.

Claim: **Fubini-Study form is Kähler.** Indeed, $d\omega|_x$ is a $U(n)$-invariant 3-form on $\mathbb{C}^n$, but such a form must vanish, because $-\operatorname{Id} \in U(n)$

REMARK: **The same argument works for all symmetric spaces.**

DEFINITION: **An almost complex submanifold** $X \subset M$ of an almost complex manifold (M, I) is a smooth submanifold which satisfies $I(TX) \subset TX$.

EXERCISE: Let $X \subset M$ be an almost complex submanifold of (M, I), where I is integrable. **Prove that $(X, I|_{TX})$ is a complex manifold.**

DEFINITION: In this situation, X is called **a complex submanifold** of M.

Corollary: **Every projective manifold (complex submanifold of $\mathbb{C}P^n$) is Kähler.** Indeed, a restriction of a closed form is again closed.

Menagerie of complex geometry

Usually, in algebraic geometry one deals with projective manifolds. There are two wider classes one has to consider necessarily when studying projective ones.

1. **Moishezon manifolds** are those which are birational to projective. **Transcendence degree of a field $k(M)$ of global meromorphic function on a compact complex manifold M satisfies $k(M) \leqslant M$;** as shown by Moishezon, equality here means that M is Moishezon.

To study birational category, one has necessarily to include Moishezon manifolds. **Any Kähler Moishezon manifold is projective (Moishezon).**

2. **Small deformations of Kähler manifolds** often result in non-projective Kähler ones (even for a torus and a K3).

Fujiki class C manifolds

A class which includes Moishezon and Kähler is called **Fujiki class C**. A manifold is **Fujiki class C** if it is bimeromorphic to a Kähler manifold. As shown by Fujiki, Fujiki class C manifolds are closed under all natural operations which occur in algebraic geometry (such as taking moduli spaces or images).

A Kähler minimal model program would imply that any Kähler manifold admits a sequence of bimeromorphic fibrations with fibers which are either projective, hyperkähler or tori, hence **the class of Kähler manifolds is probably very restricted.** It is known that **a fundamental group of Kähler manifold is very special.**

By contrast, **the class of complex manifolds is extremely huge**.

Menagerie of complex geometry II

THEOREM: (Taubes, Panov-Petrunin)
For any finitely generated, finitely presented group Γ, **there exists a compact, complex 3-dimensional manifold M with $\pi_1(M) = \Gamma$.**

CONJECTURE: Let (M, I) be an almost complex manifold, $\dim_{\mathbb{C}} M \geqslant 3$. **Then I can be deformed to a complex structure.**

REMARK: (Non-)existence of a complex structure **is highly non-trivial even in the simplest cases,** such as S^6 (which is clearly almost complex).

REMARK: We know that non-Kähler complex manifolds are much more abundant than Kähler, except in complex dimension 2, where non-Kähler manifolds are a few and much better understood than projective ones. However, **it's very hard to come with examples of compact, non-Kähler complex manifolds.**

Examples of non-Kähler manifolds

These listed below (and iterated fibrations of these) are pretty much all examples known.

EXAMPLE: (Linear) Hopf manifold is $\mathbb{C}^n \backslash 0/\langle A\rangle$, where A is an invertible linear map with all eigenvalues $|\alpha_i| < 1$. It's diffeomorphic to $S^{2n-1} \times S^1$, hence non-Kähler (Kähler manifolds have even b_{2k-1} Betti numbers). It is locally conformally Kähler (LCK).

EXAMPLE: All complex subvarieties of a Hopf manifolds are LCK. For this reason, **they are non-Kähler** (Vaisman).

EXAMPLE: Twistor space is a certain $\mathbb{C}P^1$-fibration over a Riemannian 4-manifold. All such manifolds are rationally connected (connected by rational curves), **but never Kähler** except $\mathbb{C}P^3$ and flag space (Hitchin). Theorem of Taubes is proved by constructing a twistor space with prescribed fundamental group.

EXAMPLE: Homogeneous and locally homogeneous manifolds (such as nil-manifolds) are often non-Kähler.

This explains importance of LCK manifolds (defined below).

Connections

Notation: Let M be a smooth manifold, TM its tangent bundle, $\Lambda^i M$ the bundle of differential i-forms, $C^\infty M$ the smooth functions. **The space of sections of a bundle B is denoted by B.**

DEFINITION: A **connection** on a vector bundle B is a map $B \xrightarrow{\nabla} \Lambda^1 M \otimes B$ which satisfies

$$\nabla(fb) = df \otimes b + f\nabla b$$

for all $b \in B$, $f \in C^\infty M$.

REMARK: A connection ∇ on B gives a connection $B^* \xrightarrow{\nabla^*} \Lambda^1 M \otimes B^*$ on the dual bundle, by the formula

$$d(\langle b, \beta \rangle) = \langle \nabla b, \beta \rangle + \langle b, \nabla^* \beta \rangle$$

These connections are usually denoted **by the same letter ∇.**

REMARK: For any tensor bundle $\mathcal{B}_1 := B^* \otimes B^* \otimes ... \otimes B^* \otimes B \otimes B \otimes ... \otimes B$ **a connection on B defines a connection on $\mathcal{B}_1$** using the Leibniz formula:

$$\nabla(b_1 \otimes b_2) = \nabla(b_1) \otimes b_2 + b_1 \otimes \nabla(b_2).$$

Curvature of a connection

Let M be a manifold, B a bundle, $\Lambda^i M$ the differential forms, and $\nabla: B \longrightarrow B \otimes \Lambda^1 M$ a connection. We extend ∇ to $B \otimes \Lambda^i M \xrightarrow{\nabla} B \otimes \Lambda^{i+1} M$ in a natural way, using the formula

$$\nabla(b \otimes \eta) = \nabla(b) \wedge \eta + b \otimes d\eta,$$

and define **the curvature Θ_∇** of ∇ as $\nabla \circ \nabla: B \longrightarrow B \otimes \Lambda^2 M$.

CLAIM: This operator is $C^\infty M$-linear.

REMARK: We shall consider Θ_∇ as an element of $\Lambda^2 M \otimes \operatorname{End} B$, that is, an $\operatorname{End} B$-valued 2-form.

REMARK: Given vector fields $X, Y \in TM$, the curvature can be written in terms of a connection as follows

$$\Theta_\nabla(b) = \nabla_X \nabla_Y b - \nabla_Y \nabla_X B - \nabla_{[X,Y]} b.$$

CLAIM: Suppose that the structure group of B is reduced to its subgroup G, and let ∇ be a connection which preserves this reduction. This is the same as to say that the connection form takes values in $\Lambda^1 \otimes \mathfrak{g}(B)$. **Then Θ_∇ lies in $\Lambda^2 M \otimes \mathfrak{g}(B)$.**

Local systems

DEFINITION: A **local system** is a locally constant sheaf of vector spaces.

THEOREM: A local system with fiber B at $x \in M$ gives a homomorphism $\pi_1(M,x) \longrightarrow \operatorname{Aut}(B)$. **This correspondence gives an equivalence of categories.**

Proof: The etale space of a local system is a covering of M, and the monodromy map from $\pi_1(M,x)$ to permutatons of B is by construction linear.

To obtain a converse correspondence, let $\pi: \tilde M \longrightarrow M$ be the universal cover, and $X := \tilde M \times B/\pi_1(M)$ be a quotient where $\pi_1(M)$ acts on $\tilde M \times B$ diagonally. Let $y \in M$ and $U \ni y$ be a neighbourhood for which $\pi^{-1}(U)$ is a union of several copies U. Then X is a product $B \times U$. This gives a local trivialization of $\varphi: X \longrightarrow M$. **The sheaf of locally trivial sections of φ is locally trivial**, and the corresponding monodromy map is $\pi_1(M,x) \longrightarrow \operatorname{Aut}(B)$ the one we started from. ■

Flat bundles

DEFINITION: A bundle (B, ∇) is called **flat** if its curvature vanishes.

DEFINITION: A section b of (B, ∇) is called **parallel** if $\nabla(b) = 0$.

CLAIM: Let (B, ∇) be a flat bundle on M, and $\mathcal{B}$ be the sheaf of parallel sections. **Then $\mathcal{B}$ is a locally constant sheaf.**

Proof: Indeed, through each point passes of the total space of B passes a unique parallel section, which always exists locally. ■

THEOREM: This correspondence **gives an equivalence of categories** of flat bundles and local systems.

Proof: Let $\mathcal{B}$ be a local system, and $B := \mathcal{B} \otimes_{\mathbb{R}} C^\infty M$ the corresponding vector bundle. Any section of B can be written as $\sum f_i b_i$, where b_i are sections of $\mathcal{B}$, and $f_i \in C^\infty M$. Write $\nabla(\sum f_i b_i) := \sum df_i \otimes b_i$. Clearly, this connection is flat, and the corresponding sheaf of parallel sections os $\mathcal{B}$. ■

DEFINITION: Define **the B-valued de Rham differential** on $d_\nabla \Lambda^i(M) \otimes B \longrightarrow \Lambda^{i+1}(M) \otimes B$ as $d_\nabla(\eta \otimes b) := d\eta \otimes b + (-1)^{\tilde{\eta}} \eta \wedge \nabla b$. **It is easy to check that $d_\nabla^2 = 0$.**

EXERCISE: Show that **the cohomology of the complex $(\Lambda^* M \otimes B, d_\nabla)$ are equal to the cohomology of the local system $\mathcal{B} := \ker \nabla$.**

LCS manifolds

DEFINITION: Let L be an oriented real line bundle (one-dimensional bundle) on M, equipped with a flat connection, and $\omega \in \Lambda^2(M) \otimes L$ an L-valued differential form. We say that (M, ω, L) is **locally conformally symplectic** (LCS) if $d_\nabla \omega = 0$. In this situation L is called **the weight bundle** of (M, ω), or **conformal weight**.

CLAIM: An oriented real line bundle L can be smoothly trivialized.

Proof: Choose a Riemannian metric on L. Then the set of of positive unit vectors is a nowhere degenerate section of L. ■

CLAIM: Let (M, ω, L) be an LCS manifold, l a trivialization of L, and $\theta \in \Lambda^1 M$ the corresponding connection form, $\nabla(l) = l \otimes \theta$. **Then $d\omega_l = -\omega_l \wedge \theta$, where θ is a closed 1-form,** and $\omega_l \in \Lambda^2(M)$ is ω considered as a differential form after the identification $L \cong C^\infty M$ provided by l.

Proof: After identifying L and a trivial line bundle, we obtain $0 = d_\nabla(\omega) = d(\omega) + \omega \wedge \theta$. ■

LCK manifolds

REMARK: We obtained that **the following two definitions are equivalent.**

1. LCS manifold is a manifold equipped with a non-degenerate 2-form ω satisfying $d\omega = \omega \wedge \theta$, where θ is a closed 1-form.

2. LCS manifold is a manifold equipped with a non-degenerate, closed 2-form ω taking values in a flat, oriented line bundle.

DEFINITION: Let (M, I, ω) be a Hermitian manifold, $\dim_{\mathbb{C}} M > 1$. Then M is called **locally conformally Kähler** (LCK) if $d\omega = \omega \wedge \theta$, where θ is a closed 1-form, called **the Lee form**.

REMARK: Usually one silently assumes that θ is not exact. Indeed, if $\theta = d\varphi$, then $d(e^{-\varphi}\omega) = e^{-\varphi}d\omega - e^{-\varphi}\omega\wedge\theta = 0$, and $e^{-\varphi}\omega$ is Kähler. In this case (M, I, ω) is called **globally conformally Kähler**.

REMARK: As shown above, **a manifold is locally conformally Kähler iff it admits a Kähler form taking values in a positive, flat vector bundle** L**,** called **the weight bundle**.

EXERCISE: Prove that $d\theta = 0$ **follows from** $d\omega = \omega \wedge \theta$ **when** $\dim_{\mathbb{C}} M > 2$**.**

LCK manifolds and their Kähler covers

CLAIM: Let L be a local system on M, and $\pi: \tilde M \longrightarrow M$ is a universal cover. **Then $\pi^* L$ is a trivial local system.**

Proof: Indeed, $\pi_1(\tilde M) = 0$, hence all local systems on $\tilde M$ are trivial. ■

We obtain that a universal cover of an LCK manifold admits a Kähler form taking values in a trivial bundle; **this means that it is Kähler.**

REMARK: Let (M, I, ω, θ) be an LCK manifold, $\pi: \tilde M \longrightarrow M$ its universal cover, and φ a function satisfying $d\varphi = \pi^*\theta$. Then $d(e^{-\varphi}\pi^*\omega) = e^{-\varphi} d\pi^*\omega - e^{-\varphi}\pi^*\omega \wedge \pi^*\theta = 0$, hence **the form $\tilde\omega := e^{-\varphi}\pi^*\omega$ is Kähler**.

DEFINITION: Deck transform, or **monodromy maps** of a covering $\tilde M \longrightarrow M$ are elements of the group $\operatorname{Aut}_M(\tilde M)$. **When $\tilde M$ is a universal cover, one has $\operatorname{Aut}_M(\tilde M) = \pi_1(M)$.**

REMARK: A deck transform maps $\pi^*\theta$ to itself, hence it maps φ to $\varphi + C$. This implies that a deck transform maps $\tilde\omega$ to $e^C\tilde\omega$, **acting on $\tilde M$ by Kähler homotheties.**

Kähler homotheties and LCK manifolds

DEFINITION: Let (M, ω) be an LCK manifold, $\tilde M$ its Kähler cover, and $\pi_1(M) \cong \operatorname{Aut}_M(\tilde M)$ the deck transform maps. **Homothety character** is a homomorphism $\chi: \pi_1(M) \longrightarrow \mathbb{R}^{>0}$ mapping a deck transform $\gamma \in \operatorname{Aut}_M(\tilde M)$ to the number $\frac{\gamma^*(\tilde\omega)}{\tilde\omega}$.

REMARK: Let M be a complex manifold such that its universal cover $\tilde M$ is equipped with a Kähler form $\tilde\omega$, and the deck transform acts on $\tilde M$ by Kähler homotheties. Consider a local system L on M associated with the homothety character $\chi: \pi_1(M) \longrightarrow \mathbb{R}^{>0}$, and let ψ be its trivialization. For each $\gamma \in \operatorname{Aut}_M(\tilde M)$, one has $\frac{\gamma^*\psi}{\psi} = \frac{\gamma^*(\tilde\omega)}{\tilde\omega}$. Therefore, **$\psi^{-1}\tilde\omega$ is an $\operatorname{Aut}_M(\tilde M)$-form on $\tilde M$.** Denote by ω the corresponding form on M. Then

$$d\omega = d(\psi\tilde\omega) = d\psi \wedge \tilde\omega = d\log\psi \wedge \omega.$$

We obtained that the form ω satisfies $d\omega = \omega \wedge \theta$, where $\theta = d\log\psi$. This brings one more definition of LCK manifolds.

DEFINITION: An LCK manifold is a complex manifold such that its universal cover $\tilde M$ is equipped with a Kähler form $\tilde\omega$, and the deck transform acts on $\tilde M$ by Kähler homotheties.

Examples of LCK manifolds

EXAMPLE: A classical Hopf manifold is $H := \mathbb{C}^n\backslash 0/\mathbb{Z}$, where $\mathbb{Z}$ acts as a multiplication by a complex number λ, $|\lambda| > 1$.

REMARK: Its covering has a usual Kähler form, and the mapping group acts by homotheties.

OBSERVATION: H is diffeomorphic to $S^1 \times S^{2n-1}$, and fibered over $\mathbb{C}P^{n-1}$ with fiber $\mathbb{C}^*/\langle\lambda\rangle$.

OBSERVATION: For any complex submanifold $X \subset \mathbb{C}P^{n-1}$, its preimage in H is a complex manifold.

REMARK: Obviously, any complex submanifold of an LCK manifold is again LCK. **This implies that $\sigma^* X \subset H$ is an LCK manifold.**

REMARK: Next lecture I will prove that **none of these manifolds admits a Kähler form, if $n > 1$.**