Locally conformally Kähler manifolds

lecture 2: Vaisman theorem

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Complex manifolds (reminder)

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X,Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

REMARK: The commutator defines a $\mathbb{C}^\infty M$-linear map $N := \Lambda^2(T^{1,0}) \longrightarrow T^{0,1}M$, called **the Nijenhuis tensor** of I. **One can represent N as a section of $\Lambda^{2,0}(M) \otimes T^{0,1}M$.**

Kähler manifolds (reminder)

DEFINITION: An Riemannian metric g on an almost complex manifiold M is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^2(M)$ is called **the Hermitian form** of (M, I, g).

REMARK: It is $U(1)$-invariant, hence **of Hodge type (1,1)**.

DEFINITION: A complex Hermitian manifold (M, I, ω) is called **Kähler** if $d\omega = 0$. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M, and ω **the Kähler form**.

Definition: Let $M = \mathbb{C}P^n$ be a complex projective space, and g a $U(n+1)$-invariant Riemannian form. It is called **Fubini-Study form on $\mathbb{C}P^n$**. The Fubini-Study form is obtained by taking arbitrary Riemannian form and averaging with $U(n+1)$ using the Haar measure on $U(n+1)$.

EXERCISE: Prove that **the Fubini-Study form is unique** (up to a constant multiplier).

REMINDER: The Hodge decomposition in linear algebra

DEFINITION: Let (V, I) be a space equipped with a complex structure. **The Hodge decomposition** $V \otimes_{\mathbb{R}} \mathbb{C} := V^{1,0} \oplus V^{0,1}$ is defined in such a way that $V^{1,0}$ is a $\sqrt{-1}$-eigenspace of I, and $V^{0,1}$ a $-\sqrt{-1}$-eigenspace.

CLAIM: $\Lambda^*(V \oplus W) = \Lambda^*(V) \otimes \Lambda^*(W)$

REMARK: Let $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. The decomposition $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ induces $\Lambda^*_{\mathbb{C}}(V) = \Lambda^*_{\mathbb{C}}(V^{0,1}) \otimes \Lambda^*_{\mathbb{C}}(V^{1,0})$, giving

$$\Lambda^d V_{\mathbb{C}} = \bigoplus_{p+q=d} \Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}.$$

We denote $\Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}$ by $\Lambda^{p,q} V$. The resulting decomposition $\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^{p,q} V$ is called **the Hodge decomposition of the Grassmann algebra**.

REMINDER: Cauchy-Riemann equation and Hodge decomposition

The (p, q)-decomposition is defined on differential forms on complex manifold, in a similar way.

DEFINITION: Let (M, I) be a complex manifold A differential form $\eta \in \Lambda^1(M)$ is **of type (1,0)** if $I(\eta) = \sqrt{-1}\,\eta$, and **of type (0,1)** if $I(\eta) = -\sqrt{-1}\,\eta$. The corresponding vector bundles are denoted by $\Lambda^{1,0}(M)$ and $\Lambda^{0,1}(M)$.

REMARK: Cauchy-Riemann equations can be written as $df \in \Lambda^{1,0}(M)$. That is, **a function $f \in C^\infty_{\mathbb{C}}(M)$ is holomorphic if and only if $df \in \Lambda^{1,0}(M)$.**

REMARK: Let (M, I) be a complex manifold, and $z_1, ..., z_n$ holomorphic coordinate system in $U \subset M$, with z_i being holomorphic functions on U. **Then $dz_1, ..., dz_n$ generate the bundle $\Lambda^{1,0}(M)$, and $d\bar{z}_1, ..., d\bar{z}_n$ generate $\Lambda^{0,1}(M)$.**

EXERCISE: Prove this.

REMINDER: The Hodge decomposition on a complex manifold

DEFINITION: Let (M, I) be a complex manifold, $\{U_i\}$ its covering, and and $z_1, ..., z_n$ holomorphic coordinate system on each covering patch. **The bundle $\Lambda^{p,q}(M, I)$ of (p, q)-forms on (M, I)** is generated locally on each coordinate patch by monomials $dz_{i_1} \wedge dz_{i_2} \wedge ... \wedge dz_{i_p} \wedge d\bar{z}_{i_{p+1}} \wedge ... \wedge dz_{i_{p+q}}$. **The Hodge decomposition** is a decomposition of vector bundles:

$$\Lambda^d_{\mathbb{C}}(M) = \bigoplus_{p+q=d} \Lambda^{p,q}(M).$$

EXERCISE: Prove that the **de Rham differential on a complex manifold has only two Hodge components:**

$$d\left(\Lambda^{p,q}(M)\right) \subset \Lambda^{p+1,q}(M) \oplus \Lambda^{p,q+1}(M).$$

DEFINITION: Let $d = d^{0,1} + d^{1,0}$ be the Hodge decomposition of the de Rham differential on a complex manifold, $d^{0,1} : \Lambda^{p,q}(M) \longrightarrow \Lambda^{p,q+1}(M)$ and $d^{1,0} : \Lambda^{p,q}(M) \longrightarrow \Lambda^{p+1,q}(M)$. The operators $d^{0,1}$, $d^{1,0}$ are denoted $\bar{\partial}$ and ∂ and called **the Dolbeault differentials**.

EXERCISE: Show that **$\partial^2 = 0$ is equivalent to integrability of the complex structure.**

Supercommutator (reminder)

DEFINITION: A **supercommutator** of pure operators on a graded vector space is defined by a formula $\{a,b\} = ab - (-1)^{\tilde a \tilde b} ba$.

DEFINITION: A graded associative algebra is called **graded commutative** (or "supercommutative") if its supercommutator vanishes.

EXAMPLE: The Grassmann algebra is supercommutative.

DEFINITION: A graded Lie algebra (Lie superalgebra) is a graded vector space $\mathfrak g^*$ equipped with a bilinear graded map $\{\cdot,\cdot\}: \ \mathfrak g^* \times \mathfrak g^* \longrightarrow \mathfrak g^*$ which is graded anticommutative: $\{a,b\} = -(-1)^{\tilde a \tilde b}\{b,a\}$ and satisfies **the super Jacobi identity** $\{c,\{a,b\}\} = \{\{c,a\},b\} + (-1)^{\tilde a \tilde c}\{a,\{c,b\}\}$

EXAMPLE: Consider the algebra $\operatorname{End}(A^*)$ of operators on a graded vector space, with supercommutator as above. **Then $\operatorname{End}(A^*), \{\cdot,\cdot\}$ is a graded Lie algebra.**

Lemma 1: Let d be an odd element of a Lie superalgebra, satisfying $\{d,d\} = 0$, and L an even or odd element. **Then $\{\{L,d\},d\} = 0$.**

Proof: $0 = \{L,\{d,d\}\} = \{\{L,d\},d\} + (-1)^{\tilde L}\{d,\{L,d\}\} = 2\{\{L,d\},d\}$. ■

The twisted differential d^c (reminder)

DEFINITION: The **twisted differential** is defined as $d^c := IdI^{-1}$.

CLAIM: Let (M, I) be a complex manifold. **Then $\partial := \frac{d+\sqrt{-1}\, d^c}{2}$, $\overline{\partial} := \frac{d-\sqrt{-1}\, d^c}{2}$ are the Hodge components of d**, $\partial = d^{1,0}$, $\overline{\partial} = d^{0,1}$.

Proof: The Hodge components of d are expressed as $d^{1,0} = \frac{d+\sqrt{-1}\, d^c}{2}$, $d^{0,1} = \frac{d-\sqrt{-1}\, d^c}{2}$. Indeed, $I(\frac{d+\sqrt{-1}\, d^c}{2})I^{-1} = \sqrt{-1}\frac{d+\sqrt{-1}\, d^c}{2}$, hence $\frac{d+\sqrt{-1}\, d^c}{2}$ **has Hodge type (1,0);** the same argument works for $\overline{\partial}$. ∎

CLAIM: On a complex manifold, one has $d^c = [\mathcal{W}, d]$**.**

Proof: Clearly, $[\mathcal{W}, d^{1,0}] = \sqrt{-1}\, d^{1,0}$ and $[\mathcal{W}, d^{0,1}] = -\sqrt{-1}\, d^{0,1}$. Adding these equations, obtain $d^c = [\mathcal{W}, d]$.

COROLLARY: $\{d, d^c\} = \{d, \{d, \mathcal{W}\}\} = 0$ (Lemma 1).

REMARK: Clearly, $d = \partial + \overline{\partial}$, $d^c = -\sqrt{-1}\,(\partial - \overline{\partial})$, $dd^c = -d^c d = 2\sqrt{-1}\,\partial\overline{\partial}$.

Holomorphic forms

DEFINITION: A $(p,0)$-form η on a complex manifold M is called **holomorphic**, if $\overline{\partial}\eta = 0$.

DEFINITION: Let $\Omega^1 M \subset \Lambda^1 M$ be a sheaf over M generated by $f dg$, where f, g are holomorphic. This sheaf is called **the sheaf of holomorphic differentials** on M.

CLAIM: The sheaf of holomorphic p-forms coincides with $\Lambda^p_{\mathcal{O}_M}\Omega^1 M$, where $\mathcal{O}_M$ is the sheaf of holomorphic functions.

Proof: Clearly, all sections of $\Lambda^p_{\mathcal{O}_M}\Omega^1 M$ are holomorphic. Conversely, any $(p,0)$-form can be written locally as $\eta = \sum_{I=\{i_1,...,i_p\}} \alpha_I dz_{i_1} \wedge dz_{i_2} \wedge ... \wedge dz_{i_p}$, where z_i are holomorphic coordinates. Then $\overline{\partial}\eta = \sum \overline{\partial}\alpha_I dz_{i_1} \wedge dz_{i_2} \wedge ... \wedge dz_{i_p} = 0$ implies that $\overline{\partial}\alpha_I$, because $\Lambda^{p,1}(M) = \Lambda^{p,0}(M) \otimes \Lambda^{0,1}(M)$, hence for any basis e_I in $\Lambda^{p,0}(M)$ and any $\{g_I\} \in \Lambda^{0,1}(M)$,

$$\sum_I g_I \wedge e_I = 0 \Leftrightarrow \quad \text{all} \quad g_I = 0.$$

■

EXERCISE: Prove that **on a compact Kähler manifold, any holomorphic form is closed.**

Holomorphic 1-forms and first cohomology

LEMMA: Let θ be an exact holomorphic 1-form on a compact manifold. Then $\theta = 0$.

Proof: $\theta = df$, where f is a function satisfying $\overline{\partial}\eta = 0$, hence holomorphic. Then $f = const$ by maximum principle. ■

DEFINITION: A $(0, p)$-form η is called **antiholomorphic** if $\overline{\eta}$ is holomorphic.

The following result is implied by the Hodge theory.

THEOREM: Let (M, I) be a compact Kähler manifold, and $[\theta] \in H^2(M, \mathbb{C})$ is a cohomology class. **Then $[\theta]$ can be represented by a form $\theta = \theta^{1,0} + \theta^{0,1}$, where $\theta^{1,0}$ is holomorphic and $\theta^{0,1}$ antiholomorphic.**

EXERCISE: Prove this statement for compact complex curves.

Positive $(1,1)$-forms

DEFINITION: A **positive (1,1)-form** is a real (1,1)-form on a complex manifold which can be written as $\eta = \sum_i \alpha_i \theta_i \wedge I(\theta_i)$, where θ_i are real 1-forms, and α_i positive functions.

REMARK: Hermitian forms are clearly positive. Moreover, the cone of positive forms is a closure of the cone of Hermitian forms. **One may think of positive forms as of positive semi-definite Hermitian forms.**

DEFINITION: Hermitian forms are called **strictly positive**.

CLAIM: Let (M, I) be a complex manifold, and η a real (1,1)-form. Then for each 2-dimensional real subspace $W \subset T_x M$ such that $I(W) = W$, **the restriction of η to W is proportional to its volume form with non-negative coefficient.** Conversely, **if $\eta|_W$ is non-negative for all such W, the η is positive.**

Proof: A (1,1)-form is Hermitian if and only if $\eta(x, I(x)) > 0$ for each x; it is positive if and only if $\eta(x, I(x)) \geqslant 0$. ■

Mass of a positive $(1,1)$-form

DEFINITION: Let (M, I, ω) be a Hermitian manifold, $\dim_{\mathbb{C}} M = n$. **Mass** of positive (1,1)-form η is a volume form $\eta \wedge \omega^{n-1}$.

THEOREM: (normal form for a pair of Hermitian forms)
Let g be a Hermitian metric on V, h a pseudo-Hermitian form. Then there exists an orthonormal (with respect to g) basis $x_1, I(x_1), x_2, I(x_2), ..., x_n, I(x_n)$ in V^* such that $h = \sum a_i x_i \wedge I(x_i)$.

CLAIM: Let (M, I, ω) be a Hermitian manifold, $x_1, I(x_1), x_2, I(x_2), ..., x_n, I(x_n)$ an orthonormal basis in $\Lambda^1(M, \mathbb{R})$, and $\eta = \sum_i \alpha_i x_i \wedge I(x_i)$ a positive (1,1)-form (such a basis always exists because of a normal form theorem). **Then $\eta \wedge \omega^{n-1} = \sum \alpha_i \omega^n$.**

COROLLARY: Mass of a positive form is always a (not strictly) positive volume form. **A positive form vanishes if and only if its mass vanishes.**

LCK manifolds (reminder)

DEFINITION: Let (M, I, ω) be a Hermitian manifold, $\dim_{\mathbb{C}} M > 1$. Then M is called **locally conformally Kähler** (LCK) if $d\omega = \omega \wedge \theta$, where θ is a closed 1-form, called **the Lee form**.

DEFINITION: A manifold is locally conformally Kähler iff it admits a Kähler form taking values in a positive, flat vector bundle L, called **the weight bundle**.

DEFINITION: Deck transform, or **monodromy maps** of a covering $\tilde{M} \longrightarrow M$ are elements of the group $\operatorname{Aut}_M(\tilde{M})$. **When $\tilde{M}$ is a universal cover, one has $\operatorname{Aut}_M(\tilde{M}) = \pi_1(M)$.**

DEFINITION: An LCK manifold is a complex manifold such that its universal cover $\tilde{M}$ is equipped with a Kähler form $\tilde{\omega}$, and the deck transform acts on $\tilde{M}$ by Kähler homotheties.

THEOREM: These three definitions are equivalent.

Vaisman's theorem

THEOREM: Let (M, ω, θ) be a compact LCK manifold, such that θ is not cohomologous to 0. **Then M does not admit a Kähler structure.**

Proof. Step 1: Let $d\omega = \omega\wedge\theta$, $\theta' = \theta + d\varphi$. Then $d(e^\varphi\omega) = e^\varphi\omega\wedge\theta + e^\varphi\omega\wedge d\varphi = e^\varphi\omega\wedge\theta'$. This means that **we can replace the triple (M, ω, θ) by $(M, e^\varphi\omega, \theta')$ for any 1-form θ' cohomologous to θ.**

Step 2: Assume that M admits a Kähler structure. Then θ is cohomologous to a sum of a holomorphic and antiholomorphic form. Replacing ω in its conformal class as in Step 1, **we may assume that θ is a sum of a holomorphic and antiholomorphic form.**

Step 3: Then $dd^c\theta = \sqrt{-1}\, d\bar\partial\theta = 0$, giving $dd^c(\omega^{n-1}) = \omega^{n-1}\wedge\theta\wedge I(\theta)$. **Then** $0 = \int_M dd^c(\omega^{n-1}) = \int \text{Mass}(\theta\wedge I(\theta))$, **hence $\theta\wedge I(\theta) = 0$.** ■