Locally conformally Kähler manifolds

lecture 3: Vaisman manifolds

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LCK manifolds (reminder)

DEFINITION: Let (M, I, ω) be a Hermitian manifold, $\dim_{\mathbb{C}} M > 1$. Then M is called **locally conformally Kähler** (LCK) if $d\omega = \omega \wedge \theta$, where θ is a closed 1-form, called **the Lee form**.

DEFINITION: A manifold is locally conformally Kähler iff it admits a Kähler form taking values in a positive, flat vector bundle L, called **the weight bundle**.

DEFINITION: Deck transform, or **monodromy maps** of a covering $\tilde{M} \longrightarrow M$ are elements of the group $\operatorname{Aut}_M(\tilde{M})$. **When $\tilde{M}$ is a universal cover, one has $\operatorname{Aut}_M(\tilde{M}) = \pi_1(M)$.**

DEFINITION: An LCK manifold is a complex manifold such that its universal cover $\tilde{M}$ is equipped with a Kähler form $\tilde{\omega}$, and the deck transform acts on $\tilde{M}$ by Kähler homotheties.

THEOREM: These three definitions are equivalent.

Conical Kähler manifolds

DEFINITION: Let (X, g) be a Riemannian manifold, and $C(X) := X \times \mathbb{R}^{>0}$, with the metric $t^2 g + dt^2$, where t is a coordinate on $\mathbb{R}^{>0}$. Then $C(X)$ is called **Riemannian cone** of X. **Multiplicative group $\mathbb{R}^{>0}$ acts on $C(X)$ by homotheties, $(m, t) \longrightarrow (m, \lambda t)$.**

DEFINITION: Let X be a Riemannian manifold, $(M, g) = C(X) := X \times \mathbb{R}^{>0}$ its Riemannian cone, and h_λ the standard homothety action. Assume that (M, g) is equipped with a complex structure, in such a way that g is Kähler, and h_λ acts holomorphically. Then $C(X)$ is called **a conical Kähler manifold**. In this situation, X is called **Sasakian manifold**.

REMARK: A **contact manifold** is defined as a manifold X with symplectic structure on $C(X)$, and h_λ acting by homotheties. In particular, **Sasakian manifolds are contact**. **Sasakian manifold are related to contact in the same way as Kähler manifolds are related to symplectic.**

EXAMPLE: Any odd-dimensional sphere is Sasakian **(check this!).**

EXAMPLE:
(we will have a discussion of this example in the next lecture)
Let L be a positive holomorphic line bundle on a projective manifold. **Then the total space of its unit S^1-fibration in L is Sasakian.**

Vaisman manifolds

EXAMPLE: For any given $\lambda \in \mathbb{R}^{>1}$, **the quotient $C(X)/h_\lambda$ of a conical Kähler manifold is locally conformally Kähler.**

DEFINITION: A compact LCK manifold is called **structurally Vaisman**, if its is obtained as a quotient of a conical Kähler manifold $C(X)$ by $\mathbb{Z}$ acting on $C(X)$ by holomorphic homotheties.

DEFINITION: An LCK manifold (M, g, ω, θ) is called **Vaisman** if $\nabla\theta = 0$, where ∇ is the Levi-Civita connection associated with g.

THEOREM: (Structure Theorem)
A compact LCK manifold M is Vaisman if and only if it is structurally Vaisman.

REMARK: Locally, this statement is true without compactness of M. To prove it globally on M, one needs first to show that the monodromy of the weight local system is $\mathbb{Z}$; this needs compactness.

In this lecture, I will prove the local statement; a global version of the structure theorem will be proven later in the lectures.

Levi-Civita connection (reminder)

DEFINITION: The torsion of a connection $\Lambda^1 \xrightarrow{\nabla} \Lambda^1 M \otimes \Lambda^1 M$ is a map $\operatorname{Alt} \circ \nabla - d$, where $\operatorname{Alt}: \Lambda^1 M \otimes \Lambda^1 M \longrightarrow \Lambda^2 M$ is exterior multiplication. It is a map $T_\nabla: \Lambda^1 M \longrightarrow \Lambda^2 M$.

EXERCISE: Prove that torsion is a $C^\infty M$-linear.

REMARK: The dual operator $x, y \longrightarrow \nabla_x Y - \nabla_y X - [X,Y]$ is also called **the torsion of ∇**. It is a map $\Lambda^2 TM \longrightarrow TM$.

EXERCISE: Prove that these two tensors are dual.

DEFINITION: Let (M, g) be a Riemannian manifold. A connection ∇ is called **orthogonal** if $\nabla(g) = 0$. It is called **Levi-Civita** if it is torsion-free.

THEOREM: ("the main theorem of differential geometry")
For any Riemannian manifold, the Levi-Civita connection exists, and it is unique.

Vaisman structure on conical Kähler manifolds

Let us prove the implication
(Structure Vaisman) $\Rightarrow$ (Vaisman).

THEOREM: Let $(\tilde{M}, \tilde{g}, \tilde{\omega}) = X \times \mathbb{R}^{>0}$ be a conical Kähler manifold, $\mathbb{Z} = \langle \gamma \rangle$ a group acting on $\tilde{M} = C(X)$ by Kähler homotheties, and $t : C(X) \longrightarrow \mathbb{R}^{>0}$ the projection map. **Then the form $\omega := t^{-2}\tilde{\omega}$ is an LCK form on $M := \tilde{M}\langle\gamma\rangle$, its Lee form is $t^{-1}dt$, and $\nabla\theta = 0$. Here ∇ is the Levi-Civita connection on (M, g), and $g = t^{-2}\tilde{g}$.**

Proof: **$t^{-2}\tilde{g}$ is the product metric on $C(X) = X \times \mathbb{R}$, where $z = \log t$ is the coordinate on $\mathbb{R}$ and $dz = t^{-1}dt$ the unit covector.** To find θ, notice that

$$d\omega = d(t^{-2}\tilde{\omega}) = -t^{-3}dt \wedge \tilde{\omega} = -t^{-1}dt \wedge \omega = -dz \wedge \omega.$$

Then $\nabla(dz) = 0$, because it is the unit covector on the $\mathbb{R}$ component of $(C(X), g) = X \times \mathbb{R}$. ∎

Levi-Civita connection and Kähler geometry (reminder)

THEOREM: Let (M, I, g) be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) **The complex structure I is integrable, and the Hermitian form ω is closed.**

(ii) One has $\nabla(I) = 0$, where ∇ is the Levi-Civita connection.

REMARK: The implication (ii) $\Rightarrow$ (i) is clear. Indeed, $[X, Y] = \nabla_X Y - \nabla_Y X$, hence it is a (1,0)-vector field when X, Y are of type (1,0), and then I **is integrable**. Also, $d\omega = 0$, **because ∇ is torsion-free,** and $d\omega = \operatorname{Alt}(\nabla\omega)$.

The implication (i) $\Rightarrow$ (ii) is proven by the same argument as used to construct the Levi-Civita connection.

Holonomy group (reminder)

DEFINITION: (Cartan, 1923) Let (B, ∇) be a vector bundle with connection over M. For each loop γ based in $x \in M$, let $V_{\gamma,\nabla}: B|_x \longrightarrow B|_x$ be the corresponding parallel transport along the connection. The **holonomy group** of (B, ∇) is a group generated by $V_{\gamma,\nabla}$, for all loops γ. If one takes all contractible loops instead, $V_{\gamma,\nabla}$ generates **the local holonomy**, or **the restricted holonomy** group.

REMARK: A bundle is **flat** (has vanishing curvature) **if and only if its restricted holonomy vanishes.**

REMARK: If $\nabla(\varphi) = 0$ for some tensor $\varphi \in B^{\otimes i} \otimes (B^*)^{\otimes j}$, **the holonomy group preserves φ.**

DEFINITION: Holonomy of a Riemannian manifold is holonomy of its Levi-Civita connection.

EXAMPLE: Holonomy of a Riemannian manifold lies in $O(T_xM, g|_x) = O(n)$.

EXAMPLE: Holonomy of a Kähler manifold lies in $U(T_xM, g|_x, I|_x) = U(n)$.

REMARK: The holonomy group **does not depend on the choice of a point $x \in M$.**

The de Rham splitting theorem (reminder)

COROLLARY: Let M be a Riemannian manifold, and $\mathcal{H}\mathrm{ol}_0(M) \xrightarrow{\rho} \mathrm{End}(T_xM)$ a reduced holonomy representation. Suppose that ρ is reducible: $T_xM = V_1 \oplus V_2 \oplus ... \oplus V_k$. **Then $G = \mathcal{H}\mathrm{ol}_0(M)$ also splits: $G = G_1 \times G_2 \times ... \times G_k$,** with each G_i acting trivially on all V_j with $j \neq i$.

THEOREM: (de Rham) A complete, simply connected Riemannian manifold with non-irreducible holonomy **splits as a Riemannian product,** onto factors corresponding to irreducible components of the holonomy representation.

Corollary 1: Let $X \in TM$ be a vector field saisfying $\nabla X = 0$. **Then M locally splits as a Riemannian manifold: $M = M_1 \times I$,** where $I \subset \mathbb{R}$ is interval equipped with a standard metric. ■

The Lee field and conical Kähler structures

The local implication **(Vaisman) ⇒ (Structure Vaisman)** would follow locally if we prove

Proposition 1: Let (M, ω, θ) be a Vaisman manifold, and $X = \theta^\sharp$ the vector field dual to θ, called **the Lee field**. **Then X is holomorphic.**

We deduce the local form of **(Vaisman) ⇒ (Structure Vaisman)** from Proposition 1. Choose a cover $\tilde{M} \longrightarrow M$ such that the pullback of θ is exact: $\theta = d\psi$. Since M is locally a product (Corollary 1), $M = (X \times \mathbb{R}, g_0 + dt^2)$, one has $\psi = t$, and the manifold $(\tilde{M}, \psi^2 g)$ is a Riemannian cone of $X = \psi^{-1}(c)$. **To obtain that $\tilde{M}$ is a conical Kähler manifold it remains to show that the standard homotheties of the Riemannian cone act on $\tilde{M}$ holomorphically.**

However, these homotheties are obtained by exponentiation of X.

Proposition 1 will be proven later in this lecture.

Lie derivative

DEFINITION: Let $X \in TM$ be a vector field, and e^{tX} the corresponding diffeomorphism flow. For any tensor $A \in TM^{\otimes i} \otimes T^*M^{\otimes j}$, let $\operatorname{Lie}_X(A) := \frac{d}{dt}|_{t=0}(e^{tX})^*(A)$. This operation is called **the Lie derivative**.

CLAIM: The Lie derivative satisfies the following properties.

1. **Leibniz identity**. $\operatorname{Lie}_X(\alpha \otimes \beta) = \operatorname{Lie}_X(\alpha) \otimes \beta + \alpha \otimes Lie_X(\beta)$

2. **Contraction**. Let $\Pi : TM^{\otimes i} \otimes T^*M^{\otimes j} \longrightarrow TM^{\otimes i-k} \otimes T^*M^{\otimes j-k}$ denote contraction of k components of the tensor. Then $\operatorname{Lie}_X(\Pi(\alpha)) = \Pi(\operatorname{Lie}_X(\alpha))$.

3. **Differential**. $\operatorname{Lie}_X(f) = D_X(f)$ for any function $f \in C^\infty M$.

4. **Commutator:** $\operatorname{Lie}_x(Y) = [X, Y]$ for any vector field $Y \in TM$.

5. **Cartan formula:** $\operatorname{Lie}(\eta) = (d\eta) \lrcorner X + d(\eta \lrcorner X)$, for any differential form $\eta \in \Lambda^i(M)$.

Moreover, Lie_X **is uniquely determined by the properties 1-3.** ■

Killing fields

PROPOSITION: Let $g \in \mathsf{Sym}^2 T^*M$ be a Riemannian form on TM, $X \in TM$ a vector field, ∇ the Levi-Civita connection, and $h := \mathsf{Lie}_X(g)$. **Then** $h(Y,Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y)$.

Proof: By contraction property, $\mathsf{Lie}_X(g(Y,Z)) = \mathsf{Lie}_X(g)(Y,Z) + g([X,Y],Z) + g([X,Z],Y)$. Similarly, $\nabla_X(g(Y,Z)) = \nabla_X(g)(Y,Z) + g(\nabla_X Y, Z) + g(\nabla_X Z, Y)$. However, $\nabla_X(g(Y,Z)) = \mathsf{Lie}_X(g(Y,Z))$, giving

$$\begin{aligned}\mathsf{Lie}_X(g)(Y,Z) = g(\nabla_X Y, Z) + g(\nabla_X Z, Y) - g([X,Y],Z) - g([X,Z],Y) = \\ = g(\nabla_Y X, Z) + g(\nabla_Z X, Y)\end{aligned}$$

using $\nabla_X Y - [X,Y] = \nabla_Y X$, $\nabla_X Z - [X,Z] = \nabla_Z X$. ■

REMARK: A vector field which satisfies $\mathsf{Lie}_X(g) = 0$ is called **a Killing vector field.** A vector field X is Killing if and only if the diffeomorphisms e^{tX} are isometries.

Lie derivatives and Levi-Civita connection

THEOREM: Let M be a Riemannian manifold, ∇ the Levi-Civita connection, and $X \in TM$ a vector field. Consider an operator $A_X(\psi) := \nabla_X(\psi) - \operatorname{Lie}_x(\psi)$ on tensors. Then

1. **A_X is C^∞-linear, satisfies the Leibniz rule, and commutes with contraction.**

2. **On 1-forms and vector fields, $A_X = \nabla(X)$,** where $\nabla(X) \in TM \otimes \Lambda^1 M = \operatorname{End}(TM) = \operatorname{End}(T^*M)$ is understood as an endomorphism of TM and $\Lambda^1 M$.

Proof. Step 1: Linearity follows from $\nabla_X(f\psi) = f\nabla_X(\psi) + \operatorname{Lie}_X f(\psi)$ and $\operatorname{Lie}_X(f\psi) = f \operatorname{Lie}_X(\psi) + \operatorname{Lie}_X f(\psi)$, and Leibniz and contraction identity from similar identities for Lie_X and ∇.

Step 2: Then $A_X = \nabla(X)$ for $\Lambda^1 M$ would follow from a similar identity for TM.

Step 3: On vector fields, $\nabla_X Y - [X,Y] = \nabla_Y X$ because ∇ is torsion-free. ■

Levi-Civita connection and homothety action

Theorem 1: Let θ be a 1-form on a Riemannian manifold, and X the dual vector field. **Then the following are equivalent.**

(i) $\nabla(X) = \lambda\,\mathrm{Id}$.
(ii) $\nabla(\theta) = \lambda g$.
(iii) $d\theta = 0$ and $\mathrm{Lie}_X\, g = -2\lambda g$.

Proof. Step 1: $\nabla(X) = \lambda\,\mathrm{Id}$ **is clearly equivalent to** $\nabla\theta = \lambda g$ (one is obtained from another by applying g^{-1}, which is parallel).

Step 2: (ii) and (i) $\Rightarrow$ (iii): $d\theta = \mathrm{Alt}(\nabla\theta) = 0$, since g is symmetric. From $\nabla(X) = \lambda\,\mathrm{Id}$ we obtain $\nabla_X g - \mathrm{Lie}_X\, g = \lambda\,\mathrm{Id}(g) = 2\lambda g$.

Step 3: (iii) $\Rightarrow$ (i): since $\mathrm{Lie}_X\, g = -2\lambda g$ and $\nabla_X g = 0$, we obtain that $\nabla(X)(g) = 2\lambda g$. **This implies that the symmetric part of** $\nabla(X)$**, considered as a section of** $\mathrm{End}(TM)$ **is equal to** $\lambda\,\mathrm{Id}$ (the antisymmetric part acts on g trivially). To obtain the antisymmetric part, it is more convenient to replace $\nabla(X)$ by $\nabla(\theta)$. **Then the antisymmetric part of** $\nabla(\theta)$ **is equal to** $\mathrm{Alt}(\nabla(\theta)) = d\theta = 0$. ■

Remark 1: In this situation, for each tensor $\Phi \in TM^{\otimes i} \otimes T^*M^{\otimes j}$, one has $\nabla(X)(\Phi) = (i-j)\lambda\Phi$.

Lee field on Vaisman manifolds

Proposition 1: Let (M,ω,θ) be a Vaisman manifold, and $X=\theta^\sharp$ the vector field dual to θ, called **the Lee field**. **Then X is holomorphic.**

Proof. Step 1: Locally, M is a product, $M = S\times\mathbb{R}$, with the product metric. Let $\tilde M$ be a covering of M such that the pullback of θ is exact on $\tilde M$, $\theta = d\varphi$. Then $\tilde\omega := e^{-\varphi}\omega$ is a Kähler form. **The corresponding metric $\tilde g$ on $\tilde M$ a cone metric, and X acts on $(M,\tilde g)$ by homotheties.**

Step 2: Let ∇^W be the Levi-Civita connection on the Kähler manifold $(\tilde M,\tilde g,\tilde\omega)$. It is called **Weyl connection**. **Theorem 1 implies that $\nabla^W(X) = \lambda\operatorname{Id}$, for some constant λ.**

Step 3: Since $\tilde M$ is Kähler, $\nabla^W_X I = 0$. By Remark 1, $\nabla^W(X)(I) = 0$. Therefore, **Theorem 1 implies that $\operatorname{Lie}_X(I) = 0$.** This is equivalent to diffeomorphism flow e^{tX} preserving the complex structure, and hence to X being holomorphic. ■