Locally conformally Kähler manifolds

lecture 4: Sasakian manifolds

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REMINDER: The Hodge decomposition on a complex manifold

DEFINITION: Let (M, I) be a complex manifold, $\{U_i\}$ its covering, and and $z_1, ..., z_n$ holomorphic coordinate system on each covering patch. **The bundle $\Lambda^{p,q}(M, I)$ of (p, q)-forms on (M, I)** is generated locally on each coordinate patch by monomials $dz_{i_1} \wedge dz_{i_2} \wedge ... \wedge dz_{i_p} \wedge d\overline{z}_{i_{p+1}} \wedge ... \wedge dz_{i_{p+q}}$. **The Hodge decomposition** is a decomposition of vector bundles:

$$\Lambda^d_{\mathbb{C}}(M) = \bigoplus_{p+q=d} \Lambda^{p,q}(M).$$

EXERCISE: Prove that the **de Rham differential on a complex manifold has only two Hodge components:**

$$d\left(\Lambda^{p,q}(M)\right) \subset \Lambda^{p+1,q}(M) \oplus \Lambda^{p,q+1}(M).$$

DEFINITION: Let $d = d^{0,1} + d^{1,0}$ be the Hodge decomposition of the de Rham differential on a complex manifold, $d^{0,1}: \ \Lambda^{p,q}(M) \longrightarrow \Lambda^{p,q+1}(M)$ and $d^{1,0}: \ \Lambda^{p,q}(M) \longrightarrow \Lambda^{p+1,q}(M)$. The operators $d^{0,1}$, $d^{1,0}$ are denoted $\overline{\partial}$ and ∂ and called **the Dolbeault differentials**.

EXERCISE: Show that **$\partial^2 = 0$ is equivalent to integrability of the complex structure.**

The twisted differential d^c (reminder)

DEFINITION: The **twisted differential** is defined as $d^c := IdI^{-1}$.

CLAIM: Let (M, I) be a complex manifold. **Then $\partial := \frac{d+\sqrt{-1}\, d^c}{2}$, $\overline{\partial} := \frac{d-\sqrt{-1}\, d^c}{2}$ are the Hodge components of d**, $\partial = d^{1,0}$, $\overline{\partial} = d^{0,1}$.

Proof: The Hodge components of d are expressed as $d^{1,0} = \frac{d+\sqrt{-1}\, d^c}{2}$, $d^{0,1} = \frac{d-\sqrt{-1}\, d^c}{2}$. Indeed, $I(\frac{d+\sqrt{-1}\, d^c}{2})I^{-1} = \sqrt{-1}\frac{d+\sqrt{-1}\, d^c}{2}$, hence $\frac{d+\sqrt{-1}\, d^c}{2}$ **has Hodge type (1,0);** the same argument works for $\overline{\partial}$. ■

CLAIM: $\{d, d^c\} = 0$.

REMARK: Clearly, $d = \partial + \overline{\partial}$, $d^c = -\sqrt{-1}\,(\partial - \overline{\partial})$, $dd^c = -d^c d = 2\sqrt{-1}\,\partial\overline{\partial}$.

LCK manifolds (reminder)

DEFINITION: Let (M, I, ω) be a Hermitian manifold, $\dim_{\mathbb{C}} M > 1$. Then M is called **locally conformally Kähler** (LCK) if $d\omega = \omega \wedge \theta$, where θ is a closed 1-form, called **the Lee form**.

DEFINITION: **A manifold is locally conformally Kähler** iff it admits a Kähler form taking values in a positive, flat vector bundle L, called **the weight bundle**.

DEFINITION: **Deck transform**, or **monodromy maps** of a covering $\tilde{M} \longrightarrow M$ are elements of the group $\operatorname{Aut}_M(\tilde{M})$. **When $\tilde{M}$ is a universal cover, one has $\operatorname{Aut}_M(\tilde{M}) = \pi_1(M)$.**

DEFINITION: **An LCK manifold** is a complex manifold such that its universal cover $\tilde{M}$ is equipped with a Kähler form $\tilde{\omega}$, and the deck transform acts on $\tilde{M}$ by Kähler homotheties.

THEOREM: **These three definitions are equivalent.**

Holomorphic vector bundles

DEFINITION: A (smooth) **vector bundle** on a smooth manifold is a locally trivial sheaf of $C^\infty M$-modules.

DEFINITION: A **holomorphic vector bundle** on a complex manifold is a locally trivial sheaf of $\mathcal{O}_M$-modules.

REMARK: A section b of a bundle B is often denoted as $b \in B$.

CLAIM: Let B be a holomorphic vector bundle. Consider the sheaf $B_{C^\infty} := B \otimes_{\mathcal{O}_M} C^\infty M$. It is clearly locally trivial, hence **B_{C^∞} is a smooth vector bundle.**

DEFINITION: B_{C^∞} is called **a smooth vector bundle underlying B**.

A holomorphic structure operator

DEFINITION: Let $d = d^{0,1} + d^{1,0}$ be the Hodge decomposition of the de Rham differential on a complex manifold, $d^{0,1}: \Lambda^{p,q}(M) \longrightarrow \Lambda^{p,q+1}(M)$ and $d^{1,0}: \Lambda^{p,q}(M) \longrightarrow \Lambda^{p+1,q}(M)$. The operators $d^{0,1}$, $d^{1,0}$ are denoted $\overline{\partial}$ and ∂ and called **the Dolbeault differentials**.

REMARK: From $d^2 = 0$, one obtains **$\overline{\partial}^2 = 0$ and $\partial^2 = 0$.**

REMARK: The operator $\overline{\partial}$ is $\mathcal{O}_M$-linear.

DEFINITION: Let B be a holomorphic vector bundle, and $\overline{\partial}: B_{C^\infty} \longrightarrow B_{C^\infty} \otimes \Lambda^{0,1}(M)$ an operator mapping $b \otimes f$ to $b \otimes \overline{\partial} f$, where $b \in B$ is a holomorphic section, and f a smooth function. This operator is called **a holomorphic structure operator** on B. **It is correctly defined, because $\overline{\partial}$ is $\mathcal{O}_M$-linear.**

REMARK: The kernel of $\overline{\partial}$ **coincides with the set of holomorphic sections** of B.

The $\overline{\partial}$-operator on vector bundles

DEFINITION: A $\overline{\partial}$-operator on a smooth bundle is a map $V \xrightarrow{\overline{\partial}} \Lambda^{0,1}(M) \otimes V$, satisfying $\overline{\partial}(fb) = \overline{\partial}(f) \otimes b + f\overline{\partial}(b)$ for all $f \in C^\infty M, b \in V$.

REMARK: A $\overline{\partial}$-operator on B can be extended to

$$\overline{\partial} : \Lambda^{0,i}(M) \otimes V \longrightarrow \Lambda^{0,i+1}(M) \otimes V,$$

using $\overline{\partial}(\eta \otimes b) = \overline{\partial}(\eta) \otimes b + (-1)^{\tilde{\eta}} \eta \wedge \overline{\partial}(b)$, where $b \in V$ and $\eta \in \Lambda^{0,i}(M)$.

REMARK: If $\overline{\partial}$ is a holomorphic structure operator, then $\overline{\partial}^2 = 0$.

THEOREM: Let $\overline{\partial} : V \longrightarrow \Lambda^{0,1}(M) \otimes V$ be a $\overline{\partial}$-operator, satisfying $\overline{\partial}^2 = 0$. **Then $B := \ker \overline{\partial} \subset V$ is a holomorphic vector bundle of the same rank.**

REMARK: This statement is a vector bundle analogue of Newlander-Nirenberg theorem.

DEFINITION: $\overline{\partial}$-operator $\overline{\partial} : V \longrightarrow \Lambda^{0,1}(M) \otimes V$ on a smooth manifold is called **a holomorphic structure operator**, if $\overline{\partial}^2 = 0$.

Connections and holomorphic structure operators

DEFINITION: let (B, ∇) be a smooth bundle with connection and a holomorphic structure $\overline{\partial}\, B \longrightarrow \Lambda^{0,1}(M) \otimes B$. Consider a Hodge decomposition of ∇, $\nabla = \nabla^{0,1} + \nabla^{1,0}$,

$$\nabla^{0,1}: V \longrightarrow \Lambda^{0,1}(M) \otimes V, \quad \nabla^{1,0}: V \longrightarrow \Lambda^{1,0}(M) \otimes V.$$

We say that ∇ is **compatible with the holomorphic structure** if $\nabla^{0,1} = \overline{\partial}$.

DEFINITION: **An Hermitian holomorphic vector bundle** is a smooth complex vector bundle equipped with a Hermitian metric and a holomorphic structure.

DEFINITION: **A Chern connection** on a holomorphic Hermitian vector bundle is a connection compatible with the holomorphic structure and preserving the metric.

THEOREM: On any holomorphic Hermitian vector bundle, **the Chern connection exists, and is unique.**

Curvature of a holomorphic line bundle

REMARK: When speaking of a **"curvature of a holomorphic bundle",** one usually means the curvature of a Chern connection.

REMARK: Let B be a holomorphic Hermitian line bundle, and b its non-degenerate holomorphic section. Denote by η a (1,0)-form which satisfies $\nabla^{1,0}b = \eta \otimes b$. Then $d|b|^2 = \operatorname{Re} g(\nabla^{1,0}b, b) = \operatorname{Re} \eta |b|^2$. **This gives** $\nabla^{1,0}b = \frac{\partial |b|^2}{|b|^2} b = 2\partial \log |b| b$.

REMARK: Then $\Theta_B(b) = 2\bar\partial\partial \log |b| b$, **that is,** $\Theta_B = -2\partial\bar\partial \log |b|$.

REMARK: The 2-form $2\partial\bar\partial \log |b|$ **is independent from the choice of a holomorphic section** b**.** Indeed, let $b_1 = bf$, where f is a non-vanishing holomorphic function. Then

$$\partial\bar\partial \log(|b_1|^2) = 2\partial\bar\partial \log(|b|) + \partial\bar\partial \log(f\bar f) = 2\partial\bar\partial \log(|b|) + \partial\bar\partial \log f + \partial\bar\partial \log \bar f.$$

The last two terms vanish, because a logarithm of a holomorphic function is also holomorphic, and logarithm of an antiholomorphic function is antiholomorphic.

Kähler potentials and plurisubharmonic functions

DEFINITION: A real-valued smooth function on a complex manifold is called **plurisubharmonic (psh)** if the (1,1)-form $dd^c f$ is positive, and **strictly plurisubharmonic** if $dd^c f$ is an Hermitian form.

REMARK: Since $dd^c f$ is always closed, **it is also Kähler when it is strictly positive.**

DEFINITION: Let (M, I, ω) be a Kähler manifold. **Kähler potential** is a function f such that $dd^c f = \omega$.

REMARK: **Locally, Kähler potentials always exist**. This is a non-trivial theorem which follows from Poincare-Dolbeault-Grothendieck lemma.

EXAMPLE: $z \longrightarrow |z|^2$ is a Kähler potential for the usual (flat) Hermitian metric on $\mathbb{C}^n$.

Positive line bundles

DEFINITION: Let L be a holomorphic Hermitian line bundle, and $\Theta \in \Lambda^{1,1}(M)$ the curvature of its Chern connection. L is called **positive** if $\sqrt{-1}\,\Theta$ is a strictly positive form.

REMARK: In this case it is also Kähler; indeed, $dd^c \log|h|$ is closed.

EXERCISE: Prove that **the bundle $\mathcal{O}(1)$ on $\mathbb{C}P^n$ equipped with the natural $U(n+1)$-equivariant metric is positive,** and **its curvature is the Fubini-Study form.**

REMARK: Let L be a positive line bundle on M, $\operatorname{Tot}(L) \xrightarrow{\pi} M$ its total space, $\operatorname{Tot}^*(L)$ be the space of non-zero vectors in L, and ψ a function on $\operatorname{Tot}^*(L)$ defined by $\psi(v) = |v|^2$. **Then $dd^c \log\psi = \sqrt{-1}\,\pi^*\Theta_L$, where Θ_L is the curvature of L.** Indeed, on fibers of L, $\psi = |z|^2$, and $dd^c \log\psi$ vanishes.

CLAIM: The **semipositive form $dd^c \log\psi$ on $\operatorname{Tot}^*(L)$ has one zero eigenvalue** (along the fibers) and the rest is positive. ■

COROLLARY: $dd^c\psi = \sqrt{-1}\,\psi\pi^*\Theta_L + d\psi \wedge I(d\psi)$, hence **this form is strictly positive.** ■

Regular Vaisman manifolds (reminder)

THEOREM: (Kodaira theorem)
Let X be a compact complex manifold. Then **X is projective if and only if it admits a positive line bundle.**

DEFINITION: Let X be a complex manifold, and L a positive line bundle on X. Consider the $\mathbb{C}^*$-bundle Tot$^*(L)$ with the Kähler metric $\tilde{\omega} = dd^c\psi$ defined above. Fox $\lambda \in \mathbb{C}$, $|\lambda| > 1$, and let $M :=$ Tot$^*(L)/x \sim \lambda x$ be the corresponding quotient. Clearly, the map $x \longrightarrow \lambda x$ is a Kähler homothety on (Tot$^*(L), dd^c\psi$), hence M is an LCK manifold, Such a manifold is called **regular Vaisman manifold**.

REMARK: A regular Vaisman manifold is smoothly fibered on X; **the fibers are elliptic curves $\mathbb{C}^*/x \sim \lambda x$.**

REMARK: The **classical Hopf manifold** $\mathbb{C}^n \backslash 0/x \sim \lambda x$ is an example of a regular Vaisman manifold, with $X = \mathbb{C}P^{n-1}$.

REMARK: By Kodaira theorem, **all regular Vaisman manifolds admit a holomorphic embedding to classical Hopf manifolds.**

Conical Kähler manifolds (reminder)

DEFINITION: Let (X, g) be a Riemannian manifold, and $C(X) := X \times \mathbb{R}^{>0}$, with the metric $t^2 g + dt^2$, where t is a coordinate on $\mathbb{R}^{>0}$. Then $C(X)$ is called **Riemannian cone** of X. **Multiplicative group $\mathbb{R}^{>0}$ acts on $C(X)$ by homotheties, $(m, t) \longrightarrow (m, \lambda t)$.**

DEFINITION: Let (X, g) be a Riemannian manifold, $C(X) := X \times \mathbb{R}^{>0}$ its Riemannian cone, and h_λ the homothety action. Assume that (X, g) is equipped with a complex structure, in such a way that g is Kähler, and h_λ acts holomorphically. Then $C(X)$ is called **a conical Kähler manifold**. In this situation, X is called **Sasakian manifold**.

REMARK: A **contact manifold** is defined as a manifold X with symplectic structure on $C(X)$, and h_λ acting by homotheties. In particular, **Sasakian manifolds are contact**. **Sasakian manifold are related to contact in the same way as Kähler manifolds are related to symplectic.**

EXAMPLE: Any odd-dimensional sphere is Sasakian **(check this!).**

EXAMPLE: Let L be a positive holomorphic line bundle on a projective manifold. **Then the total space of its unit S^1-fibration is Sasakian.**

Contact manifolds: three equivalent definitions

All manifolds are assumed to be oriented here.

Definition 1: Let $C(S) = (S \times \mathbb{R}^{>}0)$ be a cone, equipped with the standard action $h_\lambda(x,t) = (x, \lambda t)$. Assume that $C(S)$ is equipped with a symplectic form ω such that $h_\lambda^* \omega = \lambda^2 \omega$. Then S is called **contact manifold**.

Definition 2: Let S be an odd-dimensional manifold, and $B \subset TS$ an oriented sub-bundle of codimension 1, with Frobenius form $\Lambda^2 B \xrightarrow{\Phi} TS/B$ non-degenerate. Then S is called **contact manifold**, $B \subset TS$ **the contact bundle**.

Definition 3: Let S be manifold of dimension $2k+1$, $B \subset TS$ an oriented sub-bundle of codimension 1. Assume that for any nowhere vanishing 1-form $\theta \in \Lambda^1 S$, the form $\theta \wedge (d\theta)^k$ is a non-degenerate volume form. Then (S, B) is called **a contact manifold**, and θ **a contact form**.

THEOREM: These three definitions are equivalent.

See the proof further on in this lecture.

Basic forms (reminder)

DEFINITION: Let M be a manifold, $B \subset TM$ a sub-bundle, $\theta \in \Lambda^i M$ a differential form. It is called **basic** with respect to B if for each $b \in B$, one has $\theta \lrcorner b = 0$ and $\operatorname{Lie}_b \theta = 0$.

DEFINITION: A sub-bundle $B \subset TM$ is called **involutive** if $[B, B] \subset B$.

THEOREM: "Frobenius theorem"
Let $B \subset TM$ be an involutive sub-bundle. **Then for each point $x \in M$ there exists a neighbourhood $U \ni x$ and a smooth projection $\pi : U \longrightarrow N$ such that $B = \ker \pi$.** ■

THEOREM: Let M be a manifold, $B \subset TM$ an involutive sub-bundle, $\theta \in \Lambda^i M$ a differential form. **Then the following are equivalent.**

(i) η is basic.

(ii) for any open subset $U \subset M$ and a projection $\pi : U \longrightarrow N$ such that $B = \ker \pi$, one has $\eta = \pi^* \eta'$ for some $\eta' \in \Lambda^i N$.

Contact manifolds: three equivalent definitions (proofs)

Definition 2: Let S be an odd-dimensional manifold, and $B \subset TS$ an oriented sub-bundle of codimension 1, with Frobenius form $\Lambda^2 B \xrightarrow{\Phi} TS/B$ non-degenerate. Then S is called **contact manifold**, $B \subset TS$ **the contact bundle**.

Definition 3: Let S be manifold of dimension $2k+1$, $B \subset TS$ an oriented sub-bundle of codimension 1. Assume that for any nowhere vanishing 1-form $\theta \in \Lambda^1 S$, the form $\theta \wedge (d\theta)^k$ is a non-degenerate volume form. Then (S, B) is called **a contact manifold**, and θ **a contact form**.

Proof. Step 1: (2) $\Leftrightarrow$ (3):
for each $x, y \in B$, $d\theta(x,y) = \theta([x,y]) = \Phi(x,y)$. Therefore, the Frobenius form $\Lambda^2 B \xrightarrow{\Phi} TS/B$ can be expressed as $\langle \Phi(x,y), \theta\rangle = d\theta(x,y)$. **Non-degeneracy of $\theta\wedge(d\theta)^k$ on TM is equivalent to non-degeneracy of $d\theta = \Phi$ on $B = \ker\theta$.** Therefore, $\langle \Phi(x,y), \theta\rangle = d\theta(x,y)$ is of maximal rank if and only if $\theta \wedge (d\theta)^k$ is non-degenerate.

Contact manifolds: three equivalent definitions (proofs, part two)

Definition 1: Let $C(S) = (S \times \mathbb{R}^{>}0)$ be a cone, equipped with the standard action $h_\lambda(x,t) = (x, \lambda t)$. Assume that $C(S)$ is equipped with a symplectic form ω such that $h_\lambda^* \omega = \lambda^2 \omega$. Then S is called **contact manifold**.

Step 2: (3) $\Rightarrow$ (1):
Let $M \xrightarrow{\pi} S$ be the space of positive vectors in the oriented 1-dimensional bundle $L := TS/B$, which is trivialized by the form θ, $V \in TM$ a unit vertical vector field, and $t: M \longrightarrow \mathbb{R}$ a map which associates $\theta(v)$ to a point $(s,v) \in M$, $s \in S, v \in L|_x$. Let $T := t\pi^*\theta \in \Lambda^1 M$, and let $\omega := dT$. Consider the vector field $r = tV \in TM$. Clearly, $\operatorname{Lie}_r T = 2T$, giving $\operatorname{Lie}_r dT = 2dT$. **To prove that M is a symplectic cone of S, it remains to show that dT is symplectic.**

Step 3: (3) $\Rightarrow$ (1), second part:
Since $\ker dt = \pi^* S$, any vector field $X \in TS$ can be naturally lifted to a vector field $\pi^{-1}(X) \in \ker dt \subset TM$. For each $Y := \pi^{-1}(y), x, y \in B$, one has $dT(X,Y) = T([X,Y]) = T(\pi^{-1}([x,y]))$, hence **$dT$ is non-degenerate on $\pi^{-1}(B)$.** Also, $dT \lrcorner V = T$, and $\ker T = \langle \pi^{-1} B, V\rangle$, hence **$dT$ is non-degenerate on the symplectic orthogonal complement to $\pi^{-1}B$.**

Contact manifolds: three equivalent definitions (proofs, part three)

Definition 1: Let $C(S) = (S \times \mathbb{R}^{>}0)$ be a cone, equipped with the standard action $h_\lambda(x,t) = (x, \lambda t)$. Assume that $C(S)$ is equipped with a symplectic form ω such that $h_\lambda^* \omega = \lambda^2 \omega$. Then S is called **contact manifold**.

Definition 3: Let S be manifold of dimension $2k+1$, $B \subset TS$ an oriented sub-bundle of codimension 1. Assume that for any nowhere vanishing 1-form $\theta \in \Lambda^1 S$, the form $\theta \wedge (d\theta)^k$ is a non-degenerate volume form. Then (S, B) is called **a contact manifold**, and θ **a contact form**.

Step 4: (1) $\Rightarrow$ (3):
Let $M = C(S) = S \times \mathbb{R}^{>0}$, and $t \in C^\infty M$ the standard coordinate along $\mathbb{R}^{>0}$. Consider the vector field $r := t\frac{d}{dt}$, and the form $\theta := \omega \lrcorner r$. Since $\theta \lrcorner r = 0$ and

$$\operatorname{Lie}_r t^{-1}\theta = d(t^{-1}\theta) \lrcorner r + d(\theta \lrcorner r) = t^{-1}\theta - t^{-1}\theta + d(1) = 0,$$

the form $t^{-1}\theta$ is basic with respect to the projection $C(S) \longrightarrow S$. This gives a form θ on S. Finally, $(d\theta)^{k+1}$ is non-degenerate because $d\theta$ is symplectic. **Therefore, $(d\theta)^{k+1} \lrcorner r = (k+1)(d\theta)^k \wedge \theta$ is non-degenerate on S.**

■

Reeb field

DEFINITION: A Sasakian manifold is a contact manifold S with a Riemannian structure, such that the symplectic cone $C(S)$ with its Riemannian metric is Kähler.

DEFINITION: Let S be a Sasakian manifold, ω the Kähler form on $C(S)$, and $r = t\frac{d}{dt}$ the homothety vector field. Then $\operatorname{Lie}_{Ir} t = \langle dt, Ir\rangle = 0$, hence iR is tangent to $S \subset C(S)$. This vector field (denoted by Reeb) is called **the Reeb field** of a Sasakian manifold.

REMARK: The Reeb field is dual to the contact form $\theta = \omega \lrcorner r$.

THEOREM: The Reeb field acts on a Sasakian manifold by contact isometries.

(see the next slide)

DEFINITION: A Sasakian manifold is called **regular** if the Reeb field generates a free action of S^1, **quasiregular** if all orbits of Reeb are closed, and **irregular** otherwise.

Reeb field acts by contact isometries

THEOREM: The Reeb field acts on a Sasakian manifold by contact isometries.

Proof. Step 1: Let $(C(S), \omega)$ be the cone of a Sasakian manifold with its Kähler form, and t the standard coordinate function. A **holomorphic vector field** is a vector field v such that its diffeomorphism flow e^{tv} is holomorphic. The homothety vector field $r = d\frac{d}{dt}$ is holomorphic, because $\operatorname{Lie}_r \tilde\omega = 2\tilde\omega$, $\operatorname{Lie}_r g = 2g$, giving $\operatorname{Lie}_r I = \operatorname{Lie}_r g\omega^{-1} = 0$.

Step 2: If X is a holomorphic vector field, then IX is also holomorphic. To see this, chose (locally) a Kähler metric; then $\operatorname{Lie}_X(I) = A(I)$, where $A = \nabla(X)$ acts by the formula $A(I)(v) = A(Iv) - IA(v)$. Therefore, **X is holomorphic if and only if $\nabla(X)$ is complex linear.** Since $\nabla(I) = 0$, one has $\nabla(IX) = I(\nabla(X))$, hence $\nabla(X)$ is complex linear $\Leftrightarrow$ $\nabla(IX)$ is complex linear. Then Reeb **acts on $C(X)$ holomorphically**.

Step 3: $\operatorname{Lie}_{\text{Reeb}}\omega = d(\tilde\omega \lrcorner Ir) = d(tdt) = 0$. Therefore, $\operatorname{Lie}_{\text{Reeb}}\omega = 0$. Since $\operatorname{Lie}_{\text{Reeb}} I = 0$ as well, this implies that Reeb **is Killing.**

Step 4: Contact sub-bundle $B \subset TS$ is defined as $\ker \omega \lrcorner \frac{d}{dt}$; since **the Reeb field preserves t and ω, it preserves the contact sub-bundle.** ■

Regular Sasakian manifolds

DEFINITION: A Sasakian manifold is called **regular** if the Reeb field generates a free action of S^1, **quasiregular** if all orbits of Reeb are closed, and **irregular** otherwise.

THEOREM: Let S be a regular Sasakian manifold. **Then there exists a Kähler manifold X and a positive holomorphic Hermitian line bundle L such that S is the space of unit vectors in L.**

Proof. Step 1: Let $X = S/\operatorname{Reeb}$. This quotient is well defined and smooth, because Reeb is regular. Then $X = \mathbb{C}(S)/\mathbb{C}^*$, where the C^*-action is generated by $r = t\frac{d}{dt}, I(r)$, hence holomorphic. **Therefore, X is a complex manifold** (it's a quotient of a complex manifold by holomorphic action of a Lie group)

Proof. Step 2: Since $2\omega = d\theta = d(tIdt) = dd^c(t^2)$, the function t^2 gives a Kähler potential on the cone of S. The form $dd^c \log t^2 = \frac{\omega}{t} - \frac{dt\wedge Idt}{t^2}$ vanishes on $\langle r, I(r)\rangle$ and the rest of its eigenvalues are positive. Therefore, $dd^c \log t^2$ **is basic with respect to $\langle r, I(r)\rangle$, and is equal to a pullback of a Kähler form ω_X on X.**

Proof. Step 3: Consider a holomorphic Hermitian line bundle obtained from a $\mathbb{C}^*$-bundle $C(S) \longrightarrow X$. Clearly, S is its space of unit vectors. Its curvature is expressed by $dd^c \log |v| = \omega_X$, **hence this line bundle is positive.** ■

Quasiregular Sasakian manifold: an example

EXAMPLE: Let $M = \mathbb{C}^2 \backslash 0 = C(S^3)$ considered as a conical Kähler manifold with the standard structure, and $M_1 = M/G$, where $G = \mathbb{Z}/4$ is generated by $\tau(x,y) = (y,-x)$. Since $\tau^2 = -1$, the action of G on M is free. **Therefore, M_1 is also a conical Kähler manifold.**

CLAIM: **M_1 is quasiregular, but not regular.**

Proof: Free orbits are those which satisfy $(tx,ty) \neq (y,-x)$ for each $t \in U(1) = \{\lambda \in \mathbb{C} \ \ | \ \ |\lambda| = 1\}$, $t \neq 1$. A non-free orbit gives $tx = y, ty = -x = t^2x$, hence $t = \pm\sqrt{-1}$ and $x = \pm y$. ■

REMARK: **For each quasiregular Sasakian manifold S, the quotient S/Reeb is a Kähler orbifold.** Then S is a space of unit vectors in a positive line bundle, considered in the orbifold category.