Locally conformally Kähler manifolds

lecture 5: Structure theorem for Vaisman manifolds

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LCK manifolds (reminder)

DEFINITION: Let (M, I, ω) be a Hermitian manifold, $\dim_{\mathbb{C}} M > 1$. Then M is called **locally conformally Kähler** (LCK) if $d\omega = \omega \wedge \theta$, where θ is a closed 1-form, called **the Lee form**.

DEFINITION: A manifold is locally conformally Kähler iff it admits a Kähler form taking values in a positive, flat vector bundle L, called **the weight bundle**.

DEFINITION: Deck transform, or **monodromy maps** of a covering $\tilde{M} \longrightarrow M$ are elements of the group $\operatorname{Aut}_M(\tilde{M})$. **When $\tilde{M}$ is a universal cover, one has $\operatorname{Aut}_M(\tilde{M}) = \pi_1(M)$.**

DEFINITION: An LCK manifold is a complex manifold such that its universal cover $\tilde{M}$ is equipped with a Kähler form $\tilde{\omega}$, and the deck transform acts on $\tilde{M}$ by Kähler homotheties.

THEOREM: These three definitions are equivalent.

Conical Kähler manifolds (reminder)

DEFINITION: Let (X, g) be a Riemannian manifold, and $C(X) := X \times \mathbb{R}^{>0}$, with the metric $t^2 g + dt^2$, where t is a coordinate on $\mathbb{R}^{>0}$. Then $C(X)$ is called **Riemannian cone** of X. **Multiplicative group $\mathbb{R}^{>0}$ acts on $C(X)$ by homotheties, $(m, t) \longrightarrow (m, \lambda t)$.**

DEFINITION: Let (X, g) be a Riemannian manifold, $C(X) := X \times \mathbb{R}^{>0}$ its Riemannian cone, and h_λ the homothety action. Assume that (X, g) is equipped with a complex structure, in such a way that g is Kähler, and h_λ acts holomorphically. Then $C(X)$ is called **a conical Kähler manifold**. In this situation, X is called **Sasakian manifold**.

REMARK: A **contact manifold** is defined as a manifold X with symplectic structure on $C(X)$, and h_λ acting by homotheties. In particular, **Sasakian manifolds are contact**. **Sasakian geometry is an odd-dimensional counterpart to Kähler geometry**

EXAMPLE: Let L be a positive holomorphic line bundle on a projective manifold. **Then the total space of its unit S^1-fibration is Sasakian.**

S. Sasaki, "On differentiable manifolds with certain structures which are closely related to almost contact structure", Tohoku Math. J. 2 (1960), 459-476.

Contact manifolds (reminder)

All manifolds are assumed to be oriented here.

Definition 1: Let $C(S) = (S \times \mathbb{R}^{>}0)$ be a cone, equipped with the standard action $h_\lambda(x,t) = (x, \lambda t)$. Assume that $C(S)$ is equipped with a symplectic form ω such that $h_\lambda^* \omega = \lambda^2 \omega$. Then S is called **contact manifold**.

Definition 2: Let S be an odd-dimensional manifold, and $B \subset TS$ an oriented sub-bundle of codimension 1, with Frobenius form $\Lambda^2 B \xrightarrow{\Phi} TS/B$ non-degenerate. Then S is called **contact manifold**, $B \subset TS$ **the contact bundle**.

Definition 3: Let S be manifold of dimension $2k+1$, $B \subset TS$ an oriented sub-bundle of codimension 1. Assume that for any nowhere vanishing 1-form $\theta \in \Lambda^1 S$, the form $\theta \wedge (d\theta)^k$ is a non-degenerate volume form. Then (S,B) is called **a contact manifold**, and θ **a contact form**.

THEOREM: These three definitions are equivalent.

Reeb field (reminder)

DEFINITION: A Sasakian manifold is a contact manifold S with a Riemannian structure, such that the symplectic cone $C(S)$ with its Riemannian metric is Kähler.

DEFINITION: Let S be a Sasakian manifold, ω the Kähler form on $C(S)$, and $r = t\frac{d}{dt}$ the homothety vector field. Then $\operatorname{Lie}_{Ir} t = \langle dt, Ir\rangle = 0$, hence iR is tangent to $S \subset C(S)$. This vector field (denoted by Reeb) is called **the Reeb field** of a Sasakian manifold.

REMARK: The Reeb field is dual to the contact form $\theta = \omega \lrcorner r$.

THEOREM: The Reeb field acts on a Sasakian manifold by contact isometries.

DEFINITION: A Sasakian manifold is called **regular** if the Reeb field generates a free action of S^1, **quasiregular** if all orbits of Reeb are closed, and **irregular** otherwise.

Examples of Sasakian manifolds.

Example: Let $X \subset \mathbb{C}P^n$ be a complex submanifold, and $CX \subset \mathbb{C}^{n+1} \backslash 0$ the corresponding cone. The cone CX is obviously Kähler and homogeneous, hence **the intersection $CX \cap S^{2n-1}$ is Sasakian.** This intersection is an S^1-bundle over X. This construction gives many interesting contact manifolds, including Milnor's exotic 7-spheres, which happen to be Sasakian.

REMARK: In other words, **a link of a homogeneous singularity is always Sasakian.**

REMARK: **Every quasiregular Sasakian manifold is obtained this way,** for some Kähler metric on $\mathbb{C}^{n+1}$ (Ornea-V., arXiv:math/0609617).

REMARK: All 3-dimensional Sasakian manifolds are quasiregular (H. Geiges, 1997, F. Belgun, 2000).

REMARK: **Every Sasakian manifold is diffeomorphic to a quasiregular one** (Ornea-V., arXiv:math/0306077).

REMARK: **Every regular (quasiregular) Sasakian manifold is a total space of an S^1-bundle over a Kähler manifold (orbifold).**

Vaisman manifolds

EXAMPLE: For any given $\lambda \in \mathbb{R}^{>1}$, **the quotient $C(X)/h_\lambda$ of a conical Kähler manifold is locally conformally Kähler.**

DEFINITION: An LCK manifold (M, g, ω, θ) is called **Vaisman** if $\nabla\theta = 0$, where ∇ is the Levi-Civita connection associated with g.

THEOREM: Let M be a Vaisman manifold, $\tilde{M}$ its covering; the pullback of the Lee form θ to $\tilde{M}$ is denoted by the same letter θ. Assume that $d\psi = \theta$ on $\tilde{M}$ (such ψ exists, for example, if $\tilde{M}$ is a universal cover of M). Consider the form $\tilde{\omega} := e^{-\psi}\omega$. **Then $(\tilde{M}, \tilde{\omega})$ is a Kähler manifold, isometric to a cone.**

Proof: From Lecture 3, we know that $\tilde{\omega}$ is locally a conical Kähler metric. Let $\theta^\sharp$ be the **Lee field**, dual to θ. Then $\operatorname{Lie}_{\theta^\sharp}\psi = 2\psi$, hence the space of orbits of $e^{t\theta^\sharp}$-action is identified with $S := \psi^{-1}(c)$. This gives $\tilde{M} = C(S)$. ■

Now we shall prove a global version of this result.

Structure theorem for Vaisman manifolds

THEOREM: Every Vaisman manifold is obtained as $C(X)/\mathbb{Z}$, where X is Sasakian, $\mathbb{Z} = \left\langle (x,t) \mapsto (\varphi(x), qt) \right\rangle$, $q > 1$, and φ is a Sasakian automorphism of X. Moreover, the triple (X, φ, q) is unique.

REMARK: This gives **a Riemannian submersion $M \longrightarrow S^1$** with Sasakian fibers.

Proof. Step 1: Since $\theta^\sharp$ is parallel and Killing, $M = X \times \mathbb{R}$ locally. Fix $x_0 \in M$. Then the projection $M = X \times \mathbb{R}$ to $\mathbb{R}$ is induced by $x \longrightarrow \int_{\gamma_{x_0,x}} \theta$, for $\gamma_{x_0,x}$ some path connecting x and x_0. **Therefore, $M = X \times \mathbb{R}$ whenever θ is exact.**

DEFINITION: A **monodromy group** $\mathrm{Mon}(M)$ of an LCK manifold M is the smallest group Γ such that $M = \tilde{M}/\Gamma$ and $\tilde{M}$ is Kähler.

REMARK: This is equivalent to the pullback of θ being exact.

REMARK: Monodromy group is an image of $\pi_1(M)$ in $\mathbb{R}^{>0}$ under a map associating to any $\gamma \in \pi_1(M) \subset \mathrm{Aut}(\tilde{M})$ the number $\frac{\gamma^*\tilde{\omega}}{\tilde{\omega}}$.

The proof of Structure theorem for Vaisman manifolds

Proof. Step 2: Let $\gamma_1, ..., \gamma_k \in H_1(M, \mathbb{Z})$ be generators of homology, and $\alpha_i \int_{\gamma_i} \theta$ the corresponding periods. One has a map $M \longrightarrow \mathbb{R}/\langle \alpha_1, ..., \alpha_k \rangle$, with a commutative diagram

$$\begin{array}{ccc} \tilde{M} & \longrightarrow & M \\ \downarrow & & \downarrow \\ \mathbb{R} & \longrightarrow & \mathbb{R}/\langle \alpha_1, ..., \alpha_k \rangle \end{array}$$

with vertical lines $x \longrightarrow \int_{\gamma_{x_0,x}} \theta$. **The Riemannian submersion to S^1 will be obtained if $\mathbb{R}/\langle \alpha_1, ..., \alpha_k \rangle = S^1$.**

Step 3: Let $G \subset \pi_1(M)$ be the group generated by all $\gamma \in \pi_1(M)$ such that $\int_\gamma \theta = 0$. **Then $\Gamma = \pi_1(M)/G$ is the monodromy group of M. Therefore, $\mathbb{R}/\langle \alpha_1, ..., \alpha_k \rangle = S^1 \iff \text{Mon}(M) = \mathbb{Z}$.**

Computation of the monodromy group of a Vaisman manifold

DEFINITION: Lee field on a Vaisman manifold is the vector field $\theta^\sharp$ dual to the Lee form. Since locally a Vaisman manifold is a cone over Sasakian (as shown in Lecture 3), **$\theta^\sharp$ acts on M by holomorphic isometries, and on $\tilde M$ by non-isometric homotheties.**

The following theorem finishes the proof of Structure Theorem.

THEOREM: Let (M, ω, θ) be a compact LCK manifold, and X a vector field acting on M by isometries and on $\tilde M$ by non-isometric homotheties. **Then** $\operatorname{Mon}(M) = \mathbb{Z}$**.**

This theorem is proven later today.

REMARK: Let G be a group obtained as a closure of one-parametric group e^{tX}, $t \in \mathbb{R}$. **Since X acts by isometries, G is a compact torus, $G = (S^1)^k$.**

Computation of the monodromy group, part 2

CLAIM: Let (M, ω, θ) be a compact LCK manifold, and X a vector field acting on M by isometries and on $\tilde{M}$ by non-isometric homotheties. Let $G = (S^1)^k$ be the group obtained as a closure of one-parametric group e^{tX}, $t \in \mathbb{R}$. Consider the group $\tilde{G}$ of pairs $\tilde{f} \in \operatorname{Aut}(\tilde{M})$, $f \in G$, making the following diagram commutative.

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{f}} & \tilde{M} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow{f} & M \end{array}$$

Then $\tilde{G} \cong (S^1)^{k-1} \times \mathbb{R}$.

REMARK: From this claim, the isomorphism $\operatorname{Mon}(M) = \mathbb{Z}$ follows immediately. Indeed, $\operatorname{Mon}(M) \subset \ker p : \tilde{G} \longrightarrow G$.

Computation of the monodromy group of a Vaisman manifold (part 2)

Proof of $\tilde{G} \cong (S^1)^{k-1} \times \mathbb{R}$.

Step 1: $\tilde{G}$ is a covering of G, and **the kernel of this projection is $\tilde{G} \cap$ Mon(M).**

Step 2: Let $\tilde{G}_0 \subset \tilde{G}$ be a subgroup acting on $\tilde{M}$ by isometries. Since $\tilde{G}$ acts on $\tilde{M}$ by homotheties, $\tilde{G}_0$ **has codimension 1**. Moreover, $\tilde{G}_0$ cannot intersect Mon(M) and **it maps injectively to Aut$(M) \cong (S^1)^k$.**

Step 3: We obtain that $\tilde{G}_0 \cong (S^1)^{k-1}$ (it's codimension 1).

Step 4: Since $\tilde{G}_0$ meets every component of $\tilde{G}$, it is connected. **Therefore, $\tilde{G} \cong \tilde{G}_0 \times \mathbb{R} \cong (S^1)^{k-1} \times \mathbb{R}$.** ■

Kähler potentials and plurisubharmonic functions (reminder)

DEFINITION: A real-valued smooth function on a complex manifold is called **plurisubharmonic (psh)** if the (1,1)-form $dd^c f$ is positive, and **strictly plurisubharmonic** if $dd^c f$ is an Hermitian form.

REMARK: Since $dd^c f$ is always closed, **it is also Kähler when it is strictly positive.**

DEFINITION: Let (M, I, ω) be a Kähler manifold. **Kähler potential** is a function f such that $dd^c f = \omega$.

Theorem 1: Let S be a Sasakian manifold, $C(S) = S \times \mathbb{R}^{>0}$ its cone, t the coordinate along the second variable, and $r = t\frac{d}{dt}$. **Then t^2 is a Kähler potential on $C(S)$. Moreover, the form $dd^c \log t$ vanishes on $\langle r, I(r)\rangle$ and the rest of its eigenvalues are positive.**

Proof. Step 1: $2\omega = \operatorname{Lie}_r \omega = d(I\theta) = d(tIdt) = dd^c(t^2)$ Therefore, t^2 is a Kähler potential.

Step 2: $dd^c \log t^2 = \frac{\tilde{\omega}}{t^2} - \frac{dt \wedge Idt}{t^2}$. ■

The fundamental foliation

DEFINITION: Let M be a Vaisman manifold, $\theta^\sharp$ its Lee field, and Σ a 2-dimensional real foliation generated by $\theta^\sharp, I\theta^\sharp$. It is called **the fundamental foliation** of M. Clearly, Σ is tangent to orbits of the one-parametric group of automorphisms of the covering $\tilde M$ generated by homotheties. Therefore, **Σ is a holomorphic foliaton.**

THEOREM: Let M be a compact Vaisman manifold, and $\Sigma \subset TM$ its fundamental foliation. Then

1. **Σ is independent from the choice of the Vaisman metric.**
2. **There exists a positive, exact (1,1)-form ω_0 with $\Sigma = \ker \omega_0$.**
3. **For any complex subvariety $Z \subset M$, Z is tangent to Σ.**
4. **For any compact complex subvariety $Z \subset M$, the set of smooth points of Z is Vaisman.**

Proof of (2): Let $\tilde M = C(X)$ be the conical Kähler manifold which covers M, and $\psi: \tilde M \longrightarrow \mathbb{R}$ the function satisfying $d\psi = \theta$. **Then $\omega_0 := dd^c\psi$ is a pseudo-Hermitian form which vanishes on Σ and positive on TM/Σ (Theorem 1).** Also, $\omega_0 = d(I\theta)$, hence this form is well defined on M.

The fundamental foliation (proofs)

1. Σ is independent from the choice of the Vaisman metric.
2. There exists a positive, exact (1,1)-form ω_0 with $\Sigma = \ker \omega_0$.
3. For any complex subvariety $Z \subset M$, Z is tangent to Σ.
4. For any compact complex subvariety $Z \subset M$, the set of smooth points of Z is Vaisman.

Proof of (1): The zero foliation of ω_0 is independent from the choice of the Vaisman metric. Indeed, if tere are two Vaisman structures with ω_0 and ω_0' vanishing on differenc 1-dimensional complex foliations, the sum $\omega_0 + \omega_0'$ would be positive definite. However, $\int_M \omega_0^{\dim_{\mathbb C} M}$ vanishes, because ω_0 is exact. **Since $\Sigma = \ker \omega_0$, Σ is independent from the Vaisman structure.**

Proof of (3): For any compact subvariety $X \subset M$, the integral $\int_Z \omega_0^{\dim_{\mathbb C} M}$ vanishes, because ω_0 is exact. **Therefore, $\omega_0|_{TZ}$ has one zero eigenvalue at each point of Z.** This means precisely that $\Sigma \subset TZ$ at this point.

Proof of (4): Since the Lee field is tangent ot Z, the covering $\tilde Z \subset C(S)$ is preserved by the homotheties. Therefore it is also a conical Kähler manifold. **Then $Z = \tilde Z / \mathbb Z$ is Vaisman.** ∎

Regular Vaisman manifolds

DEFINITION: A Vaisman manifold M is called **regular**, if the leaves of the fundamental foliation are orbits of the group $(S^1)^2$ freely acting on M, **quasiregular** if these leaves are compact, and **irregular** otherwise.

THEOREM: A Vaisman manifold is regular if and only if it is a smooth elliptic fibration over a projective manifold, obtained as a quotient of a total space of non-zero vectors in a positive bundle by the action of $\mathbb{Z}$ mapping v to λv, with $|\lambda| > 1$.

Proof. Step 1: Let $\tilde{M} := C(S)$ be the corresponding conical Kähler manifold. Clearly, the leaves of Σ are obtained from orbits of the Reeb field on S by the Lee field acting on $C(S)$ as a standard homothety. **Therefore, S is regular.**
Step 2: By structure theorem, $M = C(S)/\mathbb{Z}$ acting as $\langle (x,t) \mapsto (\varphi(x), qt)\rangle$, where $q > 1$, and φ is a Sasakian automorphism of X. The leaves of Σ intersect with S by a union several copies of S^1 numbered by $\langle\varphi\rangle$. **Regularity of M implies that φ has finite order, and the corresponding group acts freely on S.**
Step 3: Now, $S_1 := S/\langle\varphi\rangle$ is a regular Sasakian manifold, hence it is a space of unit vectors in a positive line bundle L over $X := S_1/\operatorname{Reeb}$.
Step 4: By construction, X is the space of leaves of Σ, hence $\tilde{M}$ is a $\mathbb{C}^*$-fibration over X. ■