Locally conformally Kähler manifolds

lecture 6: Orbifolds and cones

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NO LECTURE 24.03.2014!
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LCK manifolds (reminder)

DEFINITION: Let (M, I, ω) be a Hermitian manifold, $\dim_{\mathbb{C}} M > 1$. Then M is called **locally conformally Kähler** (LCK) if $d\omega = \omega \wedge \theta$, where θ is a closed 1-form, called **the Lee form**.

DEFINITION: A manifold is locally conformally Kähler iff it admits a Kähler form taking values in a positive, flat vector bundle L, called **the weight bundle**.

DEFINITION: Deck transform, or **monodromy maps** of a covering $\tilde{M} \longrightarrow M$ are elements of the group $\operatorname{Aut}_M(\tilde{M})$. **When $\tilde{M}$ is a universal cover, one has $\operatorname{Aut}_M(\tilde{M}) = \pi_1(M)$.**

DEFINITION: An LCK manifold is a complex manifold such that its universal cover $\tilde{M}$ is equipped with a Kähler form $\tilde{\omega}$, and the deck transform acts on $\tilde{M}$ by Kähler homotheties.

THEOREM: These three definitions are equivalent.

Conical Kähler manifolds (reminder)

DEFINITION: Let (X,g) be a Riemannian manifold, and $C(X) := X \times \mathbb{R}^{>0}$, with the metric $t^2 g + dt^2$, where t is a coordinate on $\mathbb{R}^{>0}$. Then $C(X)$ is called **Riemannian cone** of X. **Multiplicative group $\mathbb{R}^{>0}$ acts on $C(X)$ by homotheties, $(m,t) \longrightarrow (m, \lambda t)$.**

DEFINITION: Let (X,g) be a Riemannian manifold, $C(X) := X \times \mathbb{R}^{>0}$ its Riemannian cone, and h_λ the homothety action. Assume that (X,g) is equipped with a complex structure, in such a way that g is Kähler, and h_λ acts holomorphically. Then $C(X)$ is called **a conical Kähler manifold**. In this situation, X is called **Sasakian manifold**.

REMARK: A **contact manifold** is defined as a manifold X with symplectic structure on $C(X)$, and h_λ acting by homotheties. In particular, **Sasakian manifolds are contact**. **Sasakian geometry is an odd-dimensional counterpart to Kähler geometry**

EXAMPLE: Let L be a positive holomorphic line bundle on a projective manifold. **Then the total space of its unit S^1-fibration is Sasakian.**

Contact manifolds (reminder)

All manifolds are assumed to be oriented here.

Definition 1: Let $C(S) = (S \times \mathbb{R}^>0)$ be a cone, equipped with the standard action $h_\lambda(x,t) = (x, \lambda t)$. Assume that $C(S)$ is equipped with a symplectic form ω such that $h_\lambda^* \omega = \lambda^2 \omega$. Then S is called **contact manifold**.

Definition 2: Let S be an odd-dimensional manifold, and $B \subset TS$ an oriented sub-bundle of codimension 1, with Frobenius form $\Lambda^2 B \xrightarrow{\Phi} TS/B$ non-degenerate. Then S is called **contact manifold**, $B \subset TS$ **the contact bundle**.

Definition 3: Let S be manifold of dimension $2k+1$, $B \subset TS$ an oriented sub-bundle of codimension 1. Assume that for any nowhere vanishing 1-form $\theta \in \Lambda^1 S$, the form $\theta \wedge (d\theta)^k$ is a non-degenerate volume form. Then (S, B) is called **a contact manifold**, and θ **a contact form**.

THEOREM: These three definitions are equivalent.

Reeb field (reminder)

DEFINITION: A Sasakian manifold is a contact manifold S with a Riemannian structure, such that the symplectic cone $C(S)$ with its Riemannian metric is Kähler.

DEFINITION: Let S be a Sasakian manifold, ω the Kähler form on $C(S)$, and $r = t\frac{d}{dt}$ the homothety vector field. Then $\operatorname{Lie}_{Ir} t = \langle dt, Ir\rangle = 0$, hence iR is tangent to $S \subset C(S)$. This vector field (denoted by Reeb) is called **the Reeb field** of a Sasakian manifold.

REMARK: The Reeb field is dual to the contact form $\theta = \omega \lrcorner r$.

THEOREM: The Reeb field acts on a Sasakian manifold by contact isometries.

DEFINITION: A Sasakian manifold is called **regular** if the Reeb field generates a free action of S^1, **quasiregular** if all orbits of Reeb are closed, and **irregular** otherwise.

Vaisman manifolds (reminder)

EXAMPLE: For any given $\lambda \in \mathbb{R}^{>1}$, **the quotient $C(X)/h_\lambda$ of a conical Kähler manifold is locally conformally Kähler.**

DEFINITION: An LCK manifold (M, g, ω, θ) is called **Vaisman** if $\nabla\theta = 0$, where ∇ is the Levi-Civita connection associated with g.

THEOREM: Let M be a Vaisman manifold, $\tilde{M}$ its covering; the pullback of the Lee form θ to $\tilde{M}$ is denoted by the same letter θ. Assume that $d\psi = \theta$ on $\tilde{M}$ (such ψ exists, for example, if $\tilde{M}$ is a universal cover of M). Consider the form $\tilde{\omega} := e^{-\psi}\omega$. **Then $(\tilde{M}, \tilde{\omega})$ is a Kähler manifold, isometric to a cone.**

THEOREM: **Every Vaisman manifold is obtained as $C(X)/\mathbb{Z}$,** where X is Sasakian, $\mathbb{Z} = \left\langle (x,t) \mapsto (\varphi(x), qt) \right\rangle$, $q > 1$, and φ is a Sasakian automorphism of X. Moreover, the triple (X, φ, q) is unique.

Kähler potentials and plurisubharmonic functions (reminder)

DEFINITION: A real-valued smooth function on a complex manifold is called **plurisubharmonic (psh)** if the (1,1)-form $dd^c f$ is positive, and **strictly plurisubharmonic** if $dd^c f$ is an Hermitian form.

REMARK: Since $dd^c f$ is always closed, **it is also Kähler when it is strictly positive.**

DEFINITION: Let (M, I, ω) be a Kähler manifold. **Kähler potential** is a function f such that $dd^c f = \omega$.

THEOREM: Let S be a Sasakian manifold, $C(S) = S \times \mathbb{R}^{>0}$ its cone, t the coordinate along the second variable, and $r = t\frac{d}{dt}$. **Then t^2 is a Kähler potential on $C(S)$. Moreover, the form $dd^c \log t$ vanishes on $\langle r, I(r)\rangle$ and the rest of its eigenvalues are positive.**

Proof. Step 1: $2\omega = \operatorname{Lie}_r \omega = d(I\theta) = d(tIdt) = dd^c(t^2)$ Therefore, t^2 is a Kähler potential.

Step 2: $dd^c \log t^2 = \frac{\tilde{\omega}}{t^2} - \frac{dt \wedge Idt}{t^2}$. ■

The fundamental foliation

DEFINITION: Let M be a Vaisman manifold, $\theta^\sharp$ its Lee field, and Σ a 2-dimensional real foliation generated by $\theta^\sharp, I\theta^\sharp$. It is called **the fundamental foliation** of M. Clearly, Σ is tangent to orbits of the one-parametric group of automorphisms of the covering $\tilde M$ generated by homotheties. Therefore, **Σ is a holomorphic foliaton.**

THEOREM: Let M be a compact Vaisman manifold, and $\Sigma \subset TM$ its fundamental foliation. Then

1. **Σ is independent from the choice of the Vaisman metric.**
2. **There exists a positive, exact (1,1)-form ω_0 with $\Sigma = \ker \omega_0$.**
3. **For any complex subvariety $Z \subset M$, Z is tangent to Σ.**
4. **For any compact complex subvariety $Z \subset M$, the set of smooth points of Z is Vaisman.**

Proof of (2): Let $\tilde M = C(X)$ be the conical Kähler manifold which covers M, and $\psi: \ \tilde M \longrightarrow \mathbb R$ the function satisfying $d\psi = \theta$. **Then $\omega_0 := dd^c\psi$ is a pseudo-Hermitian form which vanishes on Σ and positive on TM/Σ (Theorem 1).** Also, $\omega_0 = d(I\theta)$, hence this form is well defined on M.

The fundamental foliation (proofs)

1. **Σ is independent from the choice of the Vaisman metric.**
2. **There exists a positive, exact (1,1)-form ω_0 with $\Sigma = \ker \omega_0$.**
3. **For any complex subvariety $Z \subset M$, Z is tangent to Σ.**
4. **For any compact complex subvariety $Z \subset M$, the set of smooth points of Z is Vaisman.**

Proof of (1): The zero foliation of ω_0 is independent from the choice of the Vaisman metric. Indeed, if tere are two Vaisman structures with ω_0 and ω_0' vanishing on differenc 1-dimensional complex foliations, the sum $\omega_0 + \omega_0'$ would be positive definite. However, $\int_M \omega_0^{\dim_{\mathbb C} M}$ vanishes, because ω_0 is exact. **Since $\Sigma = \ker \omega_0$, Σ is independent from the Vaisman structure.**

Proof of (3): For any compact subvariety $X \subset M$, the integral $\int_Z \omega_0^{\dim_{\mathbb C} M}$ vanishes, because ω_0 is exact. **Therefore, $\omega_0|_{TZ}$ has one zero eigenvalue at each point of Z.** This means precisely that $\Sigma \subset TZ$ at this point.

Proof of (4): Since the Lee field is tangent ot Z, the covering $\tilde Z \subset C(S)$ is preserved by the homotheties. Therefore it is also a conical Kähler manifold. **Then $Z = \tilde Z / \mathbb Z$ is Vaisman.** ■

Regular Vaisman manifolds

DEFINITION: A Vaisman manifold M is called **regular**, if the leaves of the fundamental foliation are orbits of the group $(S^1)^2$ freely acting on M, **quasiregular** if these leaves are compact, and **irregular** otherwise.

THEOREM: A Vaisman manifold is regular if and only if it is a smooth elliptic fibration over a projective manifold, obtained as a quotient of a total space of non-zero vectors in a positive bundle by the action of $\mathbb{Z}$ mapping v to λv, with $|\lambda| > 1$.

Proof. Step 1: Let $\tilde{M} := C(S)$ be the corresponding conical Kähler manifold. Clearly, the leaves of Σ are obtained from orbits of the Reeb field on S by the Lee field acting on $C(S)$ as a standard homothety. **Therefore, S is regular.**
Step 2: By structure theorem, $M = C(S)/\mathbb{Z}$ acting as $\langle (x,t) \mapsto (\varphi(x), qt)\rangle$, where $q > 1$, and φ is a Sasakian automorphism of X. The leaves of Σ intersect with S by a union several copies of S^1 numbered by $\langle\varphi\rangle$. **Regularity of M implies that φ has finite order, and the corresponding group acts freely on S.**
Step 3: Now, $S_1 := S/\langle\varphi\rangle$ is a regular Sasakian manifold, hence it is a space of unit vectors in a positive line bundle L over $X := S_1/\operatorname{Reeb}$.
Step 4: By construction, X is the space of leaves of Σ, hence $\tilde{M}$ is a $\mathbb{C}^*$-fibration over X. ■

Orbispaces

DEFINITION: Groupoid is a category with all morphisms invertible.

DEFINITION: An action of a group on a manifold is **rigid** if the set of points with trivial stabilizer is dense.

DEFINITION: An orbispace is a topological space M, equipped with a structure of a groupoid (the points of M are objects of the groupoid category), a covering $\{U_i\}$, and continuous maps $\varphi_i : V_i \longrightarrow U_i$, where each V_i is equipped with a rigid action of a finite group G_i, satisfying the following properties.

1. $\varphi_i : V_i \longrightarrow V_i/G_i = U_i$ is the quotient map.
2. For each $x \in M$ and $U_i \ni x$, the group $\operatorname{Mor}(x, x)$ is equal to the stabilizer of x in G_i.

REMARK: An orbispace is a topological space, locally obtained as a quotient, **with the quotient structure remembered via the groupoid structure.**

Orbifolds

DEFINITION: An orbifold is an orbispace $(M, \{\varphi_i : V_i \longrightarrow V_i/G_i = U_i\})$, where all V_i are diffeomorphic to open balls in $\mathbb{R}^n$.

EXAMPLE: Let $M = \mathbb{C}P^1/((x,y) \sim (x,-y))$. **This quotient is homeomorphic to $\mathbb{C}P^1$. However, it is a different orbifold** if we consider the covering induced from $\mathbb{C}P^1/G$, $G = \{\pm 1\}$ and the groupoid structure where $\operatorname{Mor}(x,x) = \operatorname{St}_G(x)$.

DEFINITION: A smooth orbifold is an orbifold M equipped with a sheaf of functions $C^\infty M$ in such a way that for each $U_i = V_i/G_i$, the corresponding ring of sections $C^\infty U_i$ is identified with a ring of G_i-invariant smooth functions on V_i.

DEFINITION: A Riemannian metric on a smooth orbifold is a G_i-invariant metric on each V_i, compatible with the gluing maps.

Complex orbifolds

DEFINITION: A complex orbifold is an orbifold M equipped with a sheaf of functions $\mathcal{O}_M$ in such a way that each V_i is an open ball in $\mathbb{C}^n$, and for each $U_i = V_i/G_i$, the corresponding ring of sections $\mathcal{O}_{U_i}$ is identified with a ring of G_i-invariant holomorphic functions on V_i.

DEFINITION: An underlying complex variety of a complex orbifold is a complex variety with the topological space M and the structure sheaf $\mathcal{O}_M$.

EXAMPLE: Let $\mathbb{C}^*$ act on $\mathbb{C}^n$ as

$$h_t(x_1, ..., x_n) = (t^{a_1}x_1, t^{a_2}x_2, ..., t^{a_n}x_n).$$

The quotient $(\mathbb{C}^n \backslash 0)/\mathbb{C}^*$ is called **weighted projective space**, and denoted $\mathbb{C}P^{n-1}(a_1, ..., a_n)$.

EXERCISE: Prove that it is an orbifold.

Projective orbifolds

DEFINITION: A projective orbifold is a complex orbifold with the underlying complex variety projective.

DEFINITION: A holomorphic vector bundle on a complex orbifold is a G_i-equivariant vector bundle on each V_i, equipped with the G_i-invariant gluing maps satisfying cocycle condition.

THEOREM: (Baily)
Let M be a compact complex orbifold equipped with a holomorphic Hermitian vector bundle L. Assume that the curvature of L is positive definite on all V_i (in this case L is called **positive**). **Then M is projective.**

W. L. Baily, On the imbedding of V-manifolds in projective spaces, *Amer. J. Math.* **79** *(1957), 403-430.*

Quotients of torus action

THEOREM: Let T^n be a compact torus acting on a manifold M with all orbits of the same dimension. **Then M/T^n is an orbifold**.

The proof is futher in these slides.

LEMMA: The set of compact subgroups of T^n is countable. ■

LEMMA: Let M be a topological space with continuous action of T^n, and $\mathrm{St}(x)$ the stabilizer of $x \in M$ in T^n. **Then the map $x \longrightarrow \mathrm{St}(x)$ is semicontinuous:** for any sequence $\{x_i\} \subset M$, $\lim_i x_i = x$, one has $\mathrm{St}(x) \supset \lim_i \mathrm{St}(x_i)$, where $\lim_i \mathrm{St}(x_i)$ **is the set of all limit points of the sequences $\{t_i\}$, $t_i \in \mathrm{St}(x_i)$.** ■

PROPOSITION: For any sequence of compact subgroup of a torus $T_i \subset T^n$, the limit $\lim_i T_i$ contains all T_i, except a finite number.

EXERCISE: Prove this!

Subgroups of a torus

PROPOSITION: For any sequence of compact subgroup of a torus $T_i \subset T^n$, the limit $\lim_i T_i$ contains all T_i, except a finite number.

Proof. Step 1: LEMMA:
Let X, Y be subsets of a metric space, and $\delta(X,Y) := \sup_{x\in X} d(x,Y)$. Fix a flat Riemannian metric on a compact torus T^n. **Then for any compact subgroup $G \subset T^n$ there exists a positive number $\varepsilon(G)$ such that $\delta(G_1, G) > \varepsilon(G)$ unless $G_1 \subset G$.**

To see this, take $\varepsilon(G) = \frac{2}{3}R$, where R is a metric diameter of a smallest circle in the decomposition $T^n/G = (S^1)^k$, where T^n/G is considered with a flat metric induced from T^n. Then $\delta(G_1, G) \leqslant \delta(0, G_1/(G_1 \cap G)) \leqslant \varepsilon(G)$.

Step 2: Each T_i is a closure of a set $\{\alpha_i, 2\alpha_i, 3\alpha_i, ...\}$, where α_i is a sufficiently general point in T_i. Then $T_\infty := \lim_i T_i$ is the set of all limit points of $\{n\alpha_m\}$, $n, m \in \mathbb{Z}^{>0}$. Therefore, for all n, m, except finitely many, $d(n\alpha_m, T_\infty) < \varepsilon(T_\infty)$, giving $\delta(T_m, T_\infty) < \varepsilon(T_\infty)$ and $T_m \subset T_\infty$. ■

Stratification associated with a torus action

COROLLARY: Let T^n be a compact torus acting on a topological space. Consider a function $x \xrightarrow{\psi} \operatorname{St}(x)$. Then there exist a stratification of M by closed strata M_i such that **Ψ is constant on a complement of M_i by smaller strata, and $\Psi(M_i) \supset \Psi(M_j)$ whenever $M_j \supset M_i$.**

Proof: Consider the set $\mathfrak{A}$ of all compact subgroups of T^n, and let $M_\alpha := \{x \in M \ \ | \ \ \Psi(x) \supset \alpha\}$, where $\alpha \in \mathfrak{A}$. By semicontinuity, M_α is closed for each α. Relation $\Psi(M_i) \supset \Psi(M_j)$ for smaller strata follows from the proposition above. ■

Quotient orbifolds

THEOREM: Let Let $G = T^n$ be a torus acting on a complex manifold G by biholomorphic maps, and M_i the corresponding stratification. Let $H_0 :=$ St(x), where x is a general point of a maximal stratum. **Then the quotient M/G is an orbifold, and for each $x \in M/G$, the corresponding group Mor(x, x) is equal to St$(x)/H_0$.**

Proof. Step 1: All orbits of G are smooth. Indeed, the Zariski tangent space to an orbit has constant dimension, because it is a quotient of the Lie algebra of G by Lie(H_0), and **a variety with Zariski tangent space of constant dimension is smooth.**

Step 2: Define **a section** of an action of G at $x \in M$ as a smooth submanifold $S \ni x$ defined locally in some neighbourhood of x, transversal to the orbit $G \cdot x$ and having complementary dimension. **Clearly, a section exists at each $x \in M$.**

Quotient orbifolds (cont.)

THEOREM: Let Let $G = T^n$ be a torus acting on a complex manifold G by biholomorphic maps, and M_i the corresponding stratification. Let $H_0 :=$ St(x), where x is a general point of a maximal stratum. **Then the quotient M/G is an orbifold, and for each $x \in M/G$, the corresponding group Mor(x,x) is equal to St$(x)/H_0$.**

Step 3: A section at x can be always chosen St(x)-invariant. To see that, chose a G-invariant metric, let $W \subset T_xM$ be an orthogonal complement of the tangent space to $G \cdot x$, and S the union of all geodesics passing through x and tangent to W,

Step 4: Let $H_x :=$ St(x). Take a tubular neighbourgood U of an orbit $G \cdot x$ given by

$$U := \bigcup_{g \in G/H_x} gS.$$

For S sufficiently small, this gives a decomposition $U = S \times (G/H_x)$. Therefore, $S \cap gS = \emptyset$ for all $g \notin H_x$. **This implies that the map $S \longrightarrow M$ is a finite quotient map, with M locally isomorphic to $S/(H_x/H_0)$.** ■

Sasakian and Vaisman manifolds and their projective orbifolds

COROLLARY: Let M be a quasiregilar Vaisman manifold, Σ its fundamental foliation, and M/Σ the quotient space. **Them $X := M/\Sigma$ is a projective orbifold.**

Proof: X is an orbifold as proven above. Since it is a quotient of a complex space by a complex group action, X is a complex orbifold. By construction, the corresponding conical Kähler manifold $\tilde{M}$ is a total space of $\mathbb{C}^*$-bundle L (in the orbifold sense). The standard local argument implies that the curvature of L gives a Kähler orbifold metric on X. Baily's theorem implies that X is projective. ■

COROLLARY: Let S be a quasiregular Sasakian manifold, and Reeb its Reeb field. **Then $X := S/\mathsf{Reeb}$ is a projective orbifold, and S is a total space of $U(1)$-bundle over X associated with a positive holomorphic line bundle.**

Proof: $S \times S^1$ is Vaisman, and the corresponding fundamental foliation is $TS^1 \times \mathsf{Reeb}$. ■

Conical Kähler structures and homotheties

Proposition 1: Let (M,ω) be a conical Kähler manifold, and X a vector field acting on M by holomorphic, non-isometric homotheties, such that IX also acts by homotheties, and e^{tX} is defined for any real t. Then

(a) $dd^c\varphi = \omega$**, where** $\varphi = |X|^2$**.**

(b) Let $S_X := \varphi^{-1}(1)$**. Then** S_X **is Sasakian, and** M **is isometric to** $C(S_X)$**.**

(c) S_X **is quasiregular if and only if the action of** X **integrates to a holomorphic** $\mathbb{C}^*$**-action.**

Proof. Step 1: Since X, IX act by homotheties, one has a character $\chi: \langle X, IX\rangle \longrightarrow \mathbb{R}$ such that $\operatorname{Lie}_Z \omega = \chi(Z)\omega$. **Replacing** X **by some linear combination of** X, IX **if necessary, we may assume that** IX **acts by isometries.** Rescaling, we may assume that $\operatorname{Lie}_X g = 2g$.

Step 2: Define $X^\flat := g(X,\cdot)$ **("the dual 1-form").** Then $dX^\flat = \operatorname{Lie}_{IX}\omega = 0$ and $2X^\flat = \operatorname{Lie}_X(X^\flat) = d\langle X, X^\flat\rangle = d|X|^2$.

Step 3: $\operatorname{Lie}_X\omega = 2\omega$, which gives $2\omega = d(\omega \lrcorner X) = d(IX^\flat) = 2dId|x|^2$ (last equation is proven in Step 2). **This proves Proposition 1 (a).**

Conical Kähler structures and homotheties (cont.)

Proposition 1: Let (M,ω) be a conical Kähler manifold, and X a vector field acting on M by holomorphic, non-isometric homotheties, such that IX also acts by homotheties, and e^{tX} is defined for any real t. Then

(a) $dd^c\varphi = \omega$, **where** $\varphi = |X|^2$.

(b) Let $S_X := \varphi^{-1}(1)$. **Then** S_X **is Sasakian, and** M **is isometric to** $C(S_X)$.

(c) S_X **is quasiregular if and only if the action of** X **integrates to a holomorphic** $\mathbb{C}^*$**-action.**

Step 4: Let $M \longrightarrow S_X$ map m to an intersection of $e^{tX}m$ and S_X. **This gives a decomposition** $M = S_X \times \mathbb{R}^{>0}$**, compatible with the conical metric on** $S_X \times \mathbb{R}^{>0} = C(S_X)$**,** as shown in the last lecture using the Vaisman manifolds local decomposition into a product.

Step 5: Let C be the group generated by e^{tX}, e^{tIX}. Clearly, $C = \mathbb{R}^{>0} \times \{e^{tIX}\}$. The Reeb orbits on S_X are orbits of e^{tIX}, **hence they are compact if and only if** $\{e^{tIX}\}$ **is compact, equivalently, iff** $C = \mathbb{C}^*$**.** ■

Conical Kähler structures and $\mathbb{C}^*$-action

REMARK: For each holomorphic isometry h of a Vaisman manifold, h lifts to a conformal automorphism of its Kähler covering. However, **a conformal automorphism of a Kähler manifold is a homothety,** because $d(f\omega) = df \wedge \omega$, and this may vanish only when $df = 0$.

Theorem 1: Let $C(S)$ be a conical Kähler manifold, h_t the corresponding homothety action, and X its vector field. **Then there exists a vector field X_1 arbitrarily close to X** acting on $C(S)$ by holomorphic homotheties, with IX_1 also acting by homotheties, **such that the action of X_1 integrates to $\mathbb{C}^*$-action on $C(S)$.**

Proof: Fix some $\lambda > 1$, and let $M := C(S)/h_\lambda$ be the corresponding Vaisman manifold, where h_t acts isometrically. Consider the Lie group $G \subset \operatorname{Iso}(M)$ obtained as the closure of $\{h_t\}$. **For each vector field $X_1 \in \operatorname{Lie}(G)$, X_1 acts on M by holomorphic isometries,** hence it acts on $C(S)$ by homotheties; non-isometrically when X_1 is sufficiently close to X.

Choosing $X' \in \operatorname{Lie}(G)$ rational and sufficiently close to X, we obtain an isometry of M which integrates to a T^2-action on M **and to non-isometric $\mathbb{C}^*$-action on its cone.** ■

REMARK: By Proposition 1, **his gives a new cone structure on $C(S)$.**

Density of quasiregular Vaisman manifolds

COROLLARY: Let $C(S)$ be a conical Kähler manifold, with S compact. **Then $C(S)$ is holomorphically isometric to a total space of non-zero sections of a positive line bundle over a projective orbifold.** ■

COROLLARY: Any compact Vaisman manifold (M, I) admits a deformation (M, I') which is quasi-regular. Moreover, I' can be chosen arbitrarily close to I.

Proof: Take the conical Kähler manifold $C(S)$, and replace the homothety vector field X by a quasiregular one X'. Then take a quotient $C(S)/\mathbb{Z}$ by $\mathbb{Z}$ acting as $e^{\lambda X'}$. ■

COROLLARY: Any compact Sasakian manifold (M, I) admits a deformation (M, I') which is quasi-regular. Moreover, I' can be chosen arbitrarily close to I. ■

NO LECTURE 24.03.2014!
NEXT LECTURE March 31.