# Locally conformally Kähler manifolds

## lecture 7: Immersion theorem

Misha Verbitsky

**HSE and IUM, Moscow**

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## LCK manifolds (reminder)

**DEFINITION:** Let $(M, I, \omega)$ be a Hermitian manifold, $\dim_{\mathbb{C}} M > 1$. Then $M$ is called **locally conformally Kähler** (LCK) if $d\omega = \omega \wedge \theta$, where $\theta$ is a closed 1-form, called **the Lee form**.

**DEFINITION: A manifold is locally conformally Kähler** iff it admits a Kähler form taking values in a positive, flat vector bundle $L$, called **the weight bundle**.

**DEFINITION: Deck transform**, or **monodromy maps** of a covering $\tilde{M} \longrightarrow M$ are elements of the group $\operatorname{Aut}_M(\tilde{M})$. **When $\tilde{M}$ is a universal cover, one has $\operatorname{Aut}_M(\tilde{M}) = \pi_1(M)$.**

**DEFINITION: An LCK manifold** is a complex manifold such that its universal cover $\tilde{M}$ is equipped with a Kähler form $\tilde{\omega}$, and the deck transform acts on $\tilde{M}$ by Kähler homotheties.

**THEOREM: These three definitions are equivalent.**

## Conical Kähler manifolds (reminder)

**DEFINITION:** Let $(X, g)$ be a Riemannian manifold, and $C(X) := X \times \mathbb{R}^{>0}$, with the metric $t^2 g + dt^2$, where $t$ is a coordinate on $\mathbb{R}^{>0}$. Then $C(X)$ is called **Riemannian cone** of $X$. **Multiplicative group $\mathbb{R}^{>0}$ acts on $C(X)$ by homotheties, $(m, t) \longrightarrow (m, \lambda t)$.**

**DEFINITION:** Let $(X, g)$ be a Riemannian manifold, $C(X) := X \times \mathbb{R}^{>0}$ its Riemannian cone, and $h_\lambda$ the homothety action. Assume that $(X, g)$ is equipped with a complex structure, in such a way that $g$ is Kähler, and $h_\lambda$ acts holomorphically. Then $C(X)$ is called **a conical Kähler manifold**. In this situation, $X$ is called **Sasakian manifold**.

**REMARK:** A **contact manifold** is defined as a manifold $X$ with symplectic structure on $C(X)$, and $h_\lambda$ acting by homotheties. In particular, **Sasakian manifolds are contact**. **Sasakian geometry is an odd-dimensional counterpart to Kähler geometry**

**EXAMPLE:** Let $L$ be a positive holomorphic line bundle on a projective manifold. **Then the total space of its unit $S^1$-fibration is Sasakian.**

## Reeb field (reminder)

**DEFINITION: A Sasakian manifold** is a contact manifold $S$ with a Riemannian structure, such that the symplectic cone $C(S)$ with its Riemannian metric is Kähler.

**DEFINITION:** Let $S$ be a Sasakian manifold, $\omega$ the Kähler form on $C(S)$, and $r = t\frac{d}{dt}$ the homothety vector field. Then $\operatorname{Lie}_{Ir} t = \langle dt, Ir\rangle = 0$, hence $iR$ is tangent to $S \subset C(S)$. This vector field (denoted by Reeb) is called **the Reeb field** of a Sasakian manifold.

**REMARK: The Reeb field is dual to the contact form $\theta = \omega \lrcorner r$.**

**THEOREM: The Reeb field acts on a Sasakian manifold by contact isometries.**

**DEFINITION:** A Sasakian manifold is called **regular** if the Reeb field generates a free action of $S^1$, **quasiregular** if all orbits of Reeb are closed, and **irregular** otherwise.

## Vaisman manifolds (reminder)

**EXAMPLE:** For any given $\lambda \in \mathbb{R}^{>1}$, **the quotient $C(X)/h_\lambda$ of a conical Kähler manifold is locally conformally Kähler.**

**DEFINITION:** An LCK manifold $(M, g, \omega, \theta)$ is called **Vaisman** if $\nabla\theta = 0$, where $\nabla$ is the Levi-Civita connection associated with $g$.

**THEOREM:** Let $M$ be a Vaisman manifold, $\tilde{M}$ its covering; the pullback of the Lee form $\theta$ to $\tilde{M}$ is denoted by the same letter $\theta$. Assume that $d\psi = \theta$ on $\tilde{M}$ (such $\psi$ exists, for example, if $\tilde{M}$ is a universal cover of $M$). Consider the form $\tilde{\omega} := e^{-\psi}\omega$. **Then $(\tilde{M}, \tilde{\omega})$ is a Kähler manifold, isometric to a cone.**

**THEOREM: Every Vaisman manifold is obtained as $C(X)/\mathbb{Z}$,** where $X$ is Sasakian, $\mathbb{Z} = \left\langle (x,t) \mapsto (\varphi(x), qt) \right\rangle$, $q > 1$, and $\varphi$ is a Sasakian automorphism of $X$. Moreover, the triple $(X, \varphi, q)$ is unique.

## Kähler potentials and plurisubharmonic functions (reminder)

**DEFINITION:** A real-valued smooth function on a complex manifold is called **plurisubharmonic (psh)** if the (1,1)-form $dd^c f$ is positive, and **strictly plurisubharmonic** if $dd^c f$ is an Hermitian form.

**REMARK:** Since $dd^c f$ is always closed, **it is also Kähler when it is strictly positive.**

**DEFINITION:** Let $(M, I, \omega)$ be a Kähler manifold. **Kähler potential** is a function $f$ such that $dd^c f = \omega$.

**THEOREM:** Let $S$ be a Sasakian manifold, $C(S) = S \times \mathbb{R}^{>0}$ its cone, $t$ the coordinate along the second variable, and $r = t\frac{d}{dt}$. **Then $t^2$ is a Kähler potential on $C(S)$. Moreover, the form $dd^c \log t$ vanishes on $\langle r, I(r)\rangle$ and the rest of its eigenvalues are positive.**

## The fundamental foliation (reminder)

**DEFINITION:** Let $M$ be a Vaisman manifold, $\theta^\sharp$ its Lee field, and $\Sigma$ a 2-dimensional real foliation generated by $\theta^\sharp, I\theta^\sharp$. It is called **the fundamental foliation** of $M$. Clearly, $\Sigma$ is tangent to orbits of the one-parametric group of automorphisms of the covering $\tilde M$ generated by homotheties. Therefore, **$\Sigma$ is a holomorphic foliaton.**

**THEOREM:** Let $M$ be a compact Vaisman manifold, and $\Sigma \subset TM$ its fundamental foliation. Then

1. **$\Sigma$ is independent from the choice of the Vaisman metric.**
2. **There exists a positive, exact (1,1)-form $\omega_0$ with $\Sigma = \ker \omega_0$.**
3. **For any complex subvariety $Z \subset M$, $Z$ is tangent to $\Sigma$.**
4. **For any compact complex subvariety $Z \subset M$, the set of smooth points of $Z$ is Vaisman.**

**DEFINITION:** A Vaisman manifold $M$ is called **regular**, if the leaves of the fundamental foliation are orbits of the group $(S^1)^2$ freely acting on $M$, **quasiregular** if these leaves are compact, and **irregular** otherwise.

## Orbispaces (reminder)

**DEFINITION: Groupoid** is a category with all morphisms invertible.

**DEFINITION:** An action of a group on a manifold is **rigid** if the set of points with trivial stabilizer is dense.

**DEFINITION: An orbispace** is a topological space $M$, equipped with a structure of a groupoid (the points of $M$ are objects of the groupoid category), a covering $\{U_i\}$, and continuous maps $\varphi_i : V_i \longrightarrow U_i$, where each $V_i$ is equipped with a rigid action of a finite group $G_i$, satisfying the following properties.

1. $\varphi_i : V_i \longrightarrow V_i/G_i = U_i$ is the quotient map.
2. For each $x \in M$ and $U_i \ni x$, the group $\operatorname{Mor}(x, x)$ is equal to the stabilizer of $x$ in $G_i$.

**REMARK:** An orbispace is a topological space, locally obtained as a quotient, **with the quotient structure remembered via the groupoid structure.**

## Orbifolds (reminder)

**DEFINITION: An orbifold** is an orbispace $(M, \{\varphi_i : \ V_i \longrightarrow V_i/G_i = U_i\})$, where all $V_i$ are diffeomorphic to open balls in $\mathbb{R}^n$.

**EXAMPLE:** Let $M = \mathbb{C}P^1/((x,y) \sim (x,-y))$. **This quotient is homeomorphic to $\mathbb{C}P^1$. However, it is a different orbifold** if we consider the covering induced from $\mathbb{C}P^1/G$, $G = \{\pm 1\}$ and the groupoid structure where $\operatorname{Mor}(x,x) = \operatorname{St}_G(x)$.

**DEFINITION: A smooth orbifold** is an orbifold $M$ equipped with a sheaf of functions $C^\infty M$ in such a way that for each $U_i = V_i/G_i$, the corresponding ring of sections $C^\infty U_i$ is identified with a ring of $G_i$-invariant smooth functions on $V_i$.

**DEFINITION: A complex orbifold** is an orbifold $M$ equipped with a sheaf of functions $\mathcal{O}_M$ in such a way that each $V_i$ is an open ball in $\mathbb{C}^n$, and for each $U_i = V_i/G_i$, the corresponding ring of sections $\mathcal{O}_{U_i}$ is identified with a ring of $G_i$-invariant holomorphic functions on $V_i$.

## Projective orbifolds (reminder)

**DEFINITION: An underlying complex variety** of a complex orbifold is a complex variety with the topological space $M$ and the structure sheaf $\mathcal{O}_M$.

**DEFINITION: A projective orbifold** is a complex orbifold with the underlying complex variety projective.

**DEFINITION: A holomorphic vector bundle** on a complex orbifold is a $G_i$-equivariant vector bundle on each $V_i$, equipped with the $G_i$-invariant gluing maps satisfying cocycle condition.

**THEOREM: (Baily)**
Let $M$ be a compact complex orbifold equipped with a holomorphic Hermitian vector bundle $L$. Assume that the curvature of $L$ is positive definite on all $V_i$ (in this case $L$ is called **positive**). **Then $M$ is projective.**

*W. L. Baily,* On the imbedding of V-manifolds in projective spaces, *Amer. J. Math.* **79** *(1957), 403-430.*

## Sasakian and Vaisman manifolds and their projective orbifolds

**THEOREM:** Let $T^n$ be a compact torus acting on a manifold $M$ with all orbits of the same dimension. **Then $M/T^n$ is an orbifold**. ■

**COROLLARY:** Let $M$ be a quasiregilar Vaisman manifold, $\Sigma$ its fundamental foliation, and $M/\Sigma$ the quotient space. **Them $X := M/\Sigma$ is a projective orbifold.**

**Proof:** Being a $T^2$-quotient, $X$ is an orbifold. Since it is a quotient of a complex space by a complex group action, $X$ is a complex orbifold. By construction, the corresponding conical Kähler manifold $\tilde{M}$ is a total space of $\mathbb{C}^*$-bundle $L$ (in the orbifold sense). The standard local argument implies that the curvature of $L$ gives a Kähler orbifold metric on $X$. Baily's theorem implies that $X$ is projective. ■

**COROLLARY:** Let $S$ be a quasiregular Sasakian manifold, and $\operatorname{Reeb}$ its Reeb field. **Then $X := S/\operatorname{Reeb}$ is a projective orbifold, and $S$ is a total space of $U(1)$-bundle over $X$ associated with a positive holomorphic line bundle.**

**Proof:** $S \times S^1$ is Vaisman, and the corresponding fundamental foliation is $TS^1 \times \operatorname{Reeb}$. ■

## Conical Kähler structures and homotheties

**Proposition 1:** Let $(M,\omega)$ be a conical Kähler manifold, and $X$ a vector field acting on $M$ by holomorphic, non-isometric homotheties, such that $IX$ also acts by homotheties, and $e^{tX}$ is defined for any real $t$. Then

**(a)** $dd^c\varphi = \omega$**, where** $\varphi = |X|^2$**.**

**(b) Let** $S_X := \varphi^{-1}(1)$**. Then** $S_X$ **is Sasakian, and** $M$ **is isometric to** $C(S_X)$**.**

**(c)** $S_X$ **is quasiregular if and only if the action of** $X$ **integrates to a holomorphic** $\mathbb{C}^*$**-action.**

**Proof. Step 1:** Since $X$, $IX$ act by homotheties, one has a character $\chi: \langle X, IX\rangle \longrightarrow \mathbb{R}$ such that $\operatorname{Lie}_Z \omega = \chi(Z)\omega$. **Replacing** $X$ **by some linear combination of** $X, IX$ **if necessary, we may assume that** $IX$ **acts by isometries.** Rescaling, we may assume that $\operatorname{Lie}_X g = 2g$.

**Step 2:** Define $X^\flat := g(X,\cdot)$ **("the dual 1-form").** Then $dX^\flat = \operatorname{Lie}_{IX}\omega = 0$ and $2X^\flat = \operatorname{Lie}_X(X^\flat) = d\langle X, X^\flat\rangle = d|X|^2$.

**Step 3:** $\operatorname{Lie}_X \omega = 2\omega$, which gives $2\omega = d(\omega \lrcorner X) = d(IX^\flat) = 2dId|x|^2$ (last equation is proven in Step 2). **This proves Proposition 1 (a).**

## Conical Kähler structures and homotheties (cont.)

**Proposition 1:** Let $(M,\omega)$ be a conical Kähler manifold, and $X$ a vector field acting on $M$ by holomorphic, non-isometric homotheties, such that $IX$ also acts by homotheties, and $e^{tX}$ is defined for any real $t$. Then

**(a)** $dd^c\varphi = \omega$, **where** $\varphi = |X|^2$**.**

**(b) Let** $S_X := \varphi^{-1}(1)$**. Then** $S_X$ **is Sasakian, and** $M$ **is isometric to** $C(S_X)$**.**

**(c)** $S_X$ **is quasiregular if and only if the action of** $X$ **integrates to a holomorphic** $\mathbb{C}^*$**-action.**

**Step 4:** Let $M \longrightarrow S_X$ map $m$ to an intersection of $e^{tX}m$ and $S_X$. **This gives a decomposition** $M = S_X \times \mathbb{R}^{>0}$**, compatible with the conical metric on** $S_X \times \mathbb{R}^{>0} = C(S_X)$**,** as shown in the last lecture using the local decomposition of Vaisman manifolds into a product of a Sasakian manifold and $\mathbb{R}$.

**Step 5:** Let $C$ be the group generated by $e^{tX}, e^{tIX}$. Clearly, $C = \mathbb{R}^{>0} \times \{e^{tIX}\}$. The Reeb orbits on $S_X$ are orbits of $e^{tIX}$, **hence they are compact if and only if** $\{e^{tIX}\}$ **is compact, equivalently, iff** $C = \mathbb{C}^*$**.** ■

## Conical Kähler structures and $\mathbb{C}^*$-action

**REMARK:** For each holomorphic isometry $h$ of a Vaisman manifold, $h$ lifts to a conformal automorphism of its Kähler covering. However, **a conformal automorphism of a Kähler manifold is a homothety,** because $d(f\omega) = df \wedge \omega$, and this may vanish only when $df = 0$.

**Theorem 1:** Let $C(S)$ be a conical Kähler manifold, $h_t$ the corresponding homothety action, and $X$ its vector field. **Then there exists a vector field $X_1$ arbitrarily close to $X$** acting on $C(S)$ by holomorphic homotheties, with $I(X_1)$ also acting by homotheties, **such that the action of $X_1$ integrates to $\mathbb{C}^*$-action on $C(S)$.**

**Proof:** Fix some $\lambda > 1$, and let $M := C(S)/h_\lambda$ be the corresponding Vaisman manifold, where $h_t$ acts isometrically. Consider the Lie group $G \subset \operatorname{Iso}(M)$ obtained as the closure of $\{h_t\}$. **For each vector field $X_1 \in \operatorname{Lie}(G)$, $X_1$ acts on $M$ by holomorphic isometries,** hence it acts on $C(S)$ by homotheties; non-isometrically when $X_1$ is sufficiently close to $X$.

Choosing $X' \in \operatorname{Lie}(G)$ rational and sufficiently close to $X$, we obtain an isometry of $M$ which integrates to a $T^2$-action on $M$ **and to non-isometric $\mathbb{C}^*$-action on its cone.** ■

**REMARK:** By Proposition 1, **his gives a new cone structure on $C(S)$.**

## Density of quasiregular Vaisman manifolds

**COROLLARY:** Let $C(S)$ be a conical Kähler manifold, with $S$ compact. **Then $C(S)$ is holomorphically isometric to a total space of non-zero sections of a positive line bundle over a projective orbifold.** ■

**COROLLARY: Any compact Vaisman manifold $(M, I)$ admits a deformation $(M, I')$ which is quasi-regular.** Moreover, $I'$ can be chosen arbitrarily close to $I$.

**Proof:** Take the conical Kähler manifold $C(S)$, and replace the homothety vector field $X$ by a quasiregular one $X'$. Then take a quotient $C(S)/\mathbb{Z}$ by $\mathbb{Z}$ acting as $e^{\lambda X'}$. ■

**COROLLARY: Any compact Sasakian manifold $(M, I)$ admits a deformation $(M, I')$ which is quasi-regular.** Moreover, $I'$ can be chosen arbitrarily close to $I$. ■

## Immersion of conical Kähler manifolds

**COROLLARY:** Let $C(S)$ be a conical Kähler manifold. **Then there exists a holomorphic immersion $C(S) \longrightarrow C(S^{2n-1})$ equivariant under homothety,** with $C(S^{2n-1}) = \mathbb{C}^n\backslash 0$ the standard (flat) cone.

**Proof:** The manifold $C(S)$ is a space of non-zero vectors in a total space of a positive line bundle $L$ over a projective orbifold $X$. By Baily's theorem, $L^N$ is very ample, and there exists an embedding $X \stackrel{j}{\hookrightarrow} \mathbb{C}P^{n-1}$ such that $L^N = l^*(\mathcal{O}(1))$. Consider a holomorphic map $\psi_0 : C(S) \longrightarrow \operatorname{Tot}(L^N)$ mapping $v$ to $v^N$. It is an $N$-sheeted covering.

Now, define $\Psi : C(S) \longrightarrow C(S^{2n-1})$ as $\Psi(v) := j(\psi_0(v))$. **Since $\psi_0$ is etale and $j$ an embedding, $\Psi$ is an immersion.** ■

## Immersion of Vaisman manifolds

**DEFINITION: A linear Hopf manifold** is a quotient of $\mathbb{C}^n \backslash 0$ by a linear automorphism with all eigenvalues $|\alpha_i| < 1$.

**COROLLARY:** Let $M$ be a quasiregular Vaisman manifold. **Then $M$ admits an immersion into a linear Hopf manifold.**

**Proof:** Let $C(S)$ be a conical Kähler covering of $M$. Consider an immersion $\Psi: C(S) \longrightarrow C(S^{2n-1})$, and let $\gamma: \mathbb{Z} \longrightarrow \operatorname{Aut}(C(S))$ be the homothety action. Since $L$ is $\gamma$-equivariant, $\gamma$ actually induces a linear automorphism $\Gamma$ of a vector space $\mathbb{C}^n = H^0(L^N)$. Since $\gamma$ uniformly decreases the norm, the eigenvalues of $\Gamma$ are all $|\alpha_i| < 1$. This gives a commutative square

$$\begin{array}{ccc} C(S) & \xrightarrow{\Psi} & C(S^{2n-1}) \\ \Big\downarrow{/\gamma} & & \Big\downarrow{/\Gamma} \\ M & \longrightarrow & (\mathbb{C}^n \backslash 0)/\langle \Gamma \rangle \end{array}$$

with the bottom arrow holomorphic immersion. ■

**REMARK:** In fact, for each Vaisman manifold **there exists an embedding into a linear Hopf manifold.**

## Kodaira stability theorem

**THEOREM: (Kodaira)** Let $\mathcal{X} \xrightarrow{\pi} B$ be a smooth, proper, holomorphic map, and $z \in B$ a point. Assume that the fiber $X_z := \pi^{-1}(z)$ is Kähler (that is, admits Kähler structure). **Then there exists a neighbourhood $W \ni z$ such that for each $y \in W$, the fiber $X_y := \pi^{-1}(y)$ is Kähler.**

**Proof. Step 1:** Consider the relative Frölicher spectral sequence

$$R^i\pi_*(\Omega^j_B\mathcal{X}) \Rightarrow R^{i+j}\pi_*(\mathbb{C}_{\mathcal{X}}) \quad (*)$$

Here $R^{i+j}\pi_*(\mathbb{C}_{\mathcal{X}})$ is the derived pushforward of a constant sheaf (that is, a graded local system over $B$ with the fibers of grading $k$ in $y \in B$ identified with $k$-th cohomology of $X_y$), and the $E_2$ term $R^i\pi_*(\Omega^j_B\mathcal{X})$ is a coherent sheaf obtained as a derived direct image of the fiberwise de Rham algebra $\Omega^j_B\mathcal{X} = \Omega^j(\mathcal{X}/B)$.

It is a relative (over $B$) version of the usual Frölicher spectral sequence $H^i(\Omega^j M) \Rightarrow H^{i+j}(M, \mathbb{C})$. This spectral sequence gives an inequality

$$\sum_{i+j=k} \dim H^i(\Omega^j X_y) \geqslant \sum_{i+j=k} \dim H^i(\Omega^j X_z) \quad (**)$$

## Kodaira stability theorem (part 2)

**Proof. Step 1:** Consider the relative Frölicher spectral sequence

$$R^i\pi_*(\Omega^j_B\mathcal{X}) \Rightarrow R^{i+j}\pi_*(\mathbb{C}_{\mathcal{X}}) \quad (*)$$

This spectral sequence gives an inequality

$$\sum_{i+j=k} \dim H^i(\Omega^j X_y) \geqslant \sum_{i+j=k} \dim H^i(\Omega^j X_z) \quad (**)$$

Since $X_z$ is Kähler, the Frölicher spectral sequence for $X_z$ degenerates in $E_2$, giving $\sum_{i+j=k} \dim H^i(\Omega^j X_z) = h^k(X_z)$. By semicontinuity,

$$\sum_{i+j=k} \dim H^i(\Omega^j X_y) \leqslant \sum_{i+j=k} \dim H^i(\Omega^j X_z)$$

in a sufficiently small neighbourhood $U$ of $z$. Comparing this with (**), we find that **rank of $H^i(\Omega^j X_y)$ is constant in $U$,** hence the inequality (**) is equality in $U$, and **the spectral sequence (*) degenerates.**

## Kodaira stability theorem (part 3)

**Step 2:** Consider the sheaf $\mathcal{H} := R^1\pi_*(\Omega^1_B\mathcal{X})$. It is a coherent sheaf with fiber $H^{1,1}(X_y)$ at each $y \in B$. From Step 1, we obtain that $\mathcal{H}$ is locally free in $U$, and generated by fiberwise closed (1,1)-forms. Let $\Lambda^{1,1}_{cl}(\mathcal{X}/B)$ be the sheaf of fiberwise closed vertical forms on $\mathcal{X}$, and $\pi_*\Lambda^{1,1}_{cl}(\mathcal{X}/B) \stackrel{\Xi}{\longrightarrow} \mathcal{H}$ the natural projection. Choose a Hermitian metric on $\mathcal{X}$, smoothly extending the Kähler metric $\omega_z$ on $X_z$, and let $\mathcal{H} \stackrel{\Xi^*}{\longrightarrow} \pi_*\Lambda^{1,1}_{cl}(\mathcal{X}/B)$ be the Hermitian conjugate map. By construction, $\Xi^*$ is an orthogonal projection of cohomology to closed (1,1)-forms along exact 2-forms. Therefore, **$\Xi^*$ maps the Kähler class $[\omega_z]$ to its harmonic representative $\omega_z$.**

**Step 3:** Let $\tilde\omega$ be a smooth section of $\mathcal{H}$ satisfying $\tilde\omega|_z = [\omega_z]$. **Then $\Xi^*(\tilde\omega)$ is a family of closed forms $\omega_y \in \Lambda^{1,1}_{cl}(X_y)$, depending smoothly on $y \in B$.** Since all eigenvalues of $\omega_z$ are positive, the same is true for $\omega_y$ for $y$ sufficiently close to $z$. However, a closed, positive $(1,1)$-form is Kähler. ■

**REMARK:** Neither Vaisman manifolds nor LCK manifolds are stable under small deformations. However, a small deformation of Vaisman manifolds in LCK. Next lecture I will define a new class of LCK manifolds, called **LCK manifolds with potential** which is stable under small deformations and contains Vaisman manfilds.