Locally conformally Kähler manifolds

lecture 8: LCK manifolds with potential

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LCK manifolds (reminder)

DEFINITION: Let (M, I, ω) be a Hermitian manifold, $\dim_{\mathbb{C}} M > 1$. Then M is called **locally conformally Kähler** (LCK) if $d\omega = \omega \wedge \theta$, where θ is a closed 1-form, called **the Lee form**.

DEFINITION: **A manifold is locally conformally Kähler** iff it admits a Kähler form taking values in a positive, flat vector bundle L, called **the weight bundle**.

DEFINITION: **Deck transform**, or **monodromy maps** of a covering $\tilde{M} \longrightarrow M$ are elements of the group $\operatorname{Aut}_M(\tilde{M})$. **When $\tilde{M}$ is a universal cover, one has $\operatorname{Aut}_M(\tilde{M}) = \pi_1(M)$.**

DEFINITION: **An LCK manifold** is a complex manifold such that its universal cover $\tilde{M}$ is equipped with a Kähler form $\tilde{\omega}$, and the deck transform acts on $\tilde{M}$ by Kähler homotheties.

THEOREM: **These three definitions are equivalent**.

Conical Kähler manifolds (reminder)

DEFINITION: Let (X, g) be a Riemannian manifold, and $C(X) := X \times \mathbb{R}^{>0}$, with the metric $t^2 g + dt^2$, where t is a coordinate on $\mathbb{R}^{>0}$. Then $C(X)$ is called **Riemannian cone** of X. **Multiplicative group $\mathbb{R}^{>0}$ acts on $C(X)$ by homotheties, $(m, t) \longrightarrow (m, \lambda t)$.**

DEFINITION: Let (X, g) be a Riemannian manifold, $C(X) := X \times \mathbb{R}^{>0}$ its Riemannian cone, and h_λ the homothety action. Assume that (X, g) is equipped with a complex structure, in such a way that g is Kähler, and h_λ acts holomorphically. Then $C(X)$ is called **a conical Kähler manifold**. In this situation, X is called **Sasakian manifold**.

REMARK: A **contact manifold** is defined as a manifold X with symplectic structure on $C(X)$, and h_λ acting by homotheties. In particular, **Sasakian manifolds are contact**. **Sasakian geometry is an odd-dimensional counterpart to Kähler geometry**

EXAMPLE: Let L be a positive holomorphic line bundle on a projective manifold. **Then the total space of its unit S^1-fibration is Sasakian.**

Vaisman manifolds (reminder)

EXAMPLE: For any given $\lambda \in \mathbb{R}^{>1}$, **the quotient $C(X)/h_\lambda$ of a conical Kähler manifold is locally conformally Kähler.**

DEFINITION: An LCK manifold (M, g, ω, θ) is called **Vaisman** if $\nabla\theta = 0$, where ∇ is the Levi-Civita connection associated with g.

THEOREM: Let M be a Vaisman manifold, $\tilde{M}$ its covering; the pullback of the Lee form θ to $\tilde{M}$ is denoted by the same letter θ. Assume that $d\psi = \theta$ on $\tilde{M}$ (such ψ exists, for example, if $\tilde{M}$ is a universal cover of M). Consider the form $\tilde{\omega} := e^{-\psi}\omega$. **Then $(\tilde{M}, \tilde{\omega})$ is a Kähler manifold, isometric to a cone.**

THEOREM: Every Vaisman manifold is obtained as $C(X)/\mathbb{Z}$, where X is Sasakian, $\mathbb{Z} = \left\langle (x,t) \mapsto (\varphi(x), qt) \right\rangle$, $q > 1$, and φ is a Sasakian automorphism of X. Moreover, the triple (X, φ, q) is unique.

Kähler potentials and plurisubharmonic functions (reminder)

DEFINITION: A real-valued smooth function on a complex manifold is called **plurisubharmonic (psh)** if the (1,1)-form $dd^c f$ is positive, and **strictly plurisubharmonic** if $dd^c f$ is an Hermitian form.

REMARK: Since $dd^c f$ is always closed, **it is also Kähler when it is strictly positive.**

DEFINITION: Let (M, I, ω) be a Kähler manifold. **Kähler potential** is a function f such that $dd^c f = \omega$.

THEOREM: Let S be a Sasakian manifold, $C(S) = S \times \mathbb{R}^{>0}$ its cone, t the coordinate along the second variable, and $r = t\frac{d}{dt}$. **Then t^2 is a Kähler potential on $C(S)$.**

Deformational stability of LCK manifolds

DEFINITION: Let $\mathfrak{A}$ be a property of compact complex manifolds, and $\mathcal{X} \xrightarrow{\pi} B$ a smooth, proper map, that is, a smooth family of compact manifolds. We say that $\mathfrak{A}$ is **stable under small deformations** if the set of all $z \in B$ such that $X_z := \pi^{-1}(z)$ has $\mathfrak{A}$ is open in B.

EXAMPLE: Property of admitting a Kähler metric is stable under small deformations ("Kodaira stability theorem"). One also says **"Kähler manifolds are stable with respect to small deformations"**, or **a small deformation of Kähler manifold is again Kähler**.

EXAMPLE: Call a complex manifold **Hermitian symplectic** if it admits a symplectic form ω such that its $(1,1)$-part is Hermitian.

EXERCISE: Prove that **a small deformation of a Hermitian symplectic manifold is again Hermitian symplectic**

OBSERVATION: **LCK manifolds are not stable with respect to small deformations** (Belgun). Also, **Vaisman manifolds are not stable with respect to small deformations.**

Deformations of Vaisman manifolds: automorphic forms and functions

Theorem 1: Let $\mathcal{X} \xrightarrow{\pi} B$ be a smooth, proper, holomorphic map, and $z \in B$ a point. Assume that the fiber $X_z := \pi^{-1}(z)$ is Vaisman (that is, admits a Vaisman metric). **Then there exists a neighbourhood $W \ni z$ such that for each $y \in W$, the fiber $X_y := \pi^{-1}(y)$ is LCK.**

For the proof, see the next slide.

DEFINITION: Let M be an LCK manifold, $(\tilde{M}, \omega)$ is Kähler covering, equipped with an action of $\pi_1(M)$, and $\chi: \pi_1(M) \longrightarrow \mathbb{R}^{>0}$ **the weight character**, or **the character of automorphy** which puts $\gamma \in \pi_1(M)$ to a number $\frac{\gamma^* \tilde{\omega}}{\tilde{\omega}}$. Its image $\Gamma \subset \mathbb{R}^{>0}$ is called **the monodromy group of an LCK manifold M**. We shall always chose $\tilde{M}$ in such a way that Γ acts on $\tilde{M}$ and $M = \tilde{M}/\Gamma$.

DEFINITION: Let M be a manifold, $\tilde{M}$ its Galois covering. A form η on $\tilde{M}$ is called **automorphic** if for any $\gamma \in \pi_1(M)$ acting on $\tilde{M}$ as usual, the form $\gamma^*\eta$ is proportional to η. The character $\chi_\eta(\gamma) := \frac{\gamma^*\tilde{\eta}}{\tilde{\eta}}$ is called **the character of automorphy** for η.

EXAMPLE: Let M be a Vaisman manifold, $\tilde{M} = C(S)$ its Kähler covering, and $\varphi = t^2$ its Kähler potential. **Then φ is an automorphic function.**

Deformations of Vaisman manifolds

Theorem 1: Let $\mathcal{X} \xrightarrow{\pi} B$ be a smooth, proper, holomorphic map, and $z \in B$ a point. Assume that the fiber $X_z := \pi^{-1}(z)$ is Vaisman (that is, admits a Vaisman metric). **Then there exists a neighbourhood $W \ni z$ such that for each $y \in W$, the fiber $X_y := \pi^{-1}(y)$ is LCK.**

Proof. Step 1: Let $\tilde{X}_z = C(S)$ be a conical Kähler covering of X_z. By Ehresmann fibration theorem, π is a locally trivial fibration. Replacing B by a sufficiently small open neighbourhood of z, **we may assume that π is trivial as a smooth fibration: $\mathcal{X} = X_z \times B$.** Consider a covering $\tilde{\mathcal{X}} \longrightarrow \mathcal{X}$ with $\tilde{\mathcal{X}} = \tilde{X}_z \times B$, and let $\tilde{X}_y$ denote the fibers of the projection $\tilde{\mathcal{X}} \xrightarrow{\tilde{\pi}} B$.

Proof. Step 2: Let φ be the automorphic Kähler potential of $\tilde{X}_z = C(S)$. Extend φ to $\tilde{\mathcal{X}} = \tilde{X}_z \times B$ using the projection $\tilde{\mathcal{X}} \longrightarrow \tilde{X}_z$. Restricting φ to $\tilde{X}_y \subset \tilde{\mathcal{X}}$, **we obtain an automorphic function φ_y on any $\tilde{X}_y$.**

Proof. Step 3: The form $dd^c\varphi_y$ is closed, automorphic and of type (1,1). Therefore, $\tilde{X}_y$ is LCK whenever the pseudo-Hermitian form $dd^c\varphi_y$ is positive definite. However, the complex structure on X_y smoothly depends on $y \in B$, hence the function $y \longrightarrow dd^c\varphi_y$ is continuous, and its eigenvalues continuously depend on $y \in B$. Therefore, **for y sufficiently close to z, these eigenvalues remain positive,** and φ_y gives an automorphic Kähler potential. ■

LCK manifolds with potential

DEFINITION: Let M be an LCK manifold, and $(\tilde M, \tilde\omega)$ its Kähler covering. It is called **LCK manifold with potential** if $\tilde M$ admits an automorphic Kähler potential $\varphi: \tilde M \longrightarrow \mathbb{R}^{>0}$, $dd^c\varphi = \tilde\omega$, which is **proper** (preimage of a compact is again compact).

EXAMPLE: Vaisman manifold is an example of an LCK manifold with potential.

REMARK: The property of being LCK with potential is stable under small deformations (Theorem 1; same proof).

REMARK: For any complex submanifold $Z \subset M$ of an LCK manifold with potential, Z is also an LCK manifold with potential.

Monodromy group of LCK manifolds with potential

PROPOSITION: Let M be an LCK manifold, $\Gamma \subset \mathbb{R}^{>0}$ the monodromy group, and $(\tilde{M}, \tilde{\omega})$ its Kähler covering, with $\tilde{M}/\Gamma = M$. Assume that $\tilde{\omega}$ admits a Γ-automorphic Kähler potential φ. **The map φ is proper if and only if $\Gamma = \mathbb{Z}$.**

Proof. $\Gamma = \mathbb{Z} \Rightarrow \varphi$ is proper:
Let γ be a generator of $\mathbb{Z}$, such that $\gamma^*\varphi = \lambda\varphi$, and $\pi : \tilde{M} \longrightarrow \tilde{M}/\Gamma = M$ the quotient map. **Then $\varphi^{-1}([1, \lambda[)$ is a fundamental domain of Γ-action.** Therefore $\pi : \varphi^{-1}([1, \sqrt{\lambda}]) \longrightarrow M$ is bijective onto its image, which is compact, hence $\varphi^{-1}([1, \sqrt{\lambda}])$ is also compact. This implies that **preimage of any closed interval is compact**.

Proof. φ is proper $\Rightarrow \Gamma = \mathbb{Z}$:
Assume $\gamma \neq \mathbb{Z}$; then **Γ is a dense subgroup of $\mathbb{R}^{>0}$.** Fix $x \in \tilde{M}$ and nonempty interval $]a, b[\subset \mathbb{R}^{>0}$, and let $\mathfrak{H} \subset \Gamma$ be the set of all $\gamma \in \Gamma$ such that $\varphi(\gamma(x)) \in]a, b[$ Since Γ is dense, $\mathfrak{H}$ is infinite. However, $\varphi(\mathfrak{H} \cdot x) \subset [a, b]$, hence **an infinite discrete set $\mathfrak{H} \cdot x$ is contained in a compact $\varphi^{-1}([a, b])$.** Contradiction! ■

Linear Hopf manifold

DEFINITION: Let $A \in \text{End}(\mathbb{C}^n)$ be an invertible linear endomorphism with all eigenvalues $|\alpha_i| < 1$. The quotient $H := (\mathbb{C}^n \backslash 0)/\langle A\rangle$ is called **a linear Hopf manifold**. When A can be diagonalized, H is called **a diagonal Hopf manifold**.

THEOREM: A linear Hopf manifold is LCK with potential.

Proof in lecture 10.

THEOREM: Let M be an LCK manifold with potential, $\dim_{\mathbb{C}} M > 2$. Then M admits a holomorphic embedding to a linear Hopf manifold.

Proof in lecture 9.

Stein manifolds.

DEFINITION: A complex variety M is called **holomorphically convex** if for any infinite discrete subset $S \subset M$, there exists a holomorphic function $f \in \mathcal{O}_M$ which is unbounded on S.

DEFINITION: A complex variety is called **Stein** if it is holomorphically convex, and has no compact complex subvarieties.

REMARK: Equivalently, **a complex variety is Stein if it admits a closed holomorphic embedding into $\mathbb{C}^n$.**

THEOREM: (K. Oka, 1942) **A complex manifold M is Stein** if and only M admits a Kähler metric with a **Kähler potential which is positive and proper** (proper = preimages of compact sets are compact).

THEOREM: (H. Cartan, 1951) **A complex variety M is Stein** if and only if for any coherent sheaf F on M, **its cohomology $H^i(F)$ vanish for all $i > 0$.**

Rossi-Andreotti-Siu theorem.

THEOREM: (Rossi 1965, Andreotti-Siu 1970)
Let M be a complex manifold with boundary, $\dim_{\mathbb{C}} M > 2$, and φ a proper Kähler potential on M, taking values in $[a, \infty[$, and equal to c in the boundary of M. **Then there exists a Stein variety M_a** with isolated singularities, obtained by gluing a compact domain to M, and it is unique. Moreover, any holomorphic function on M can be extended to M_a.

Theorem 2: Let M be an LCK manifold with potential, and $\tilde{M}$ its Kähler $\mathbb{Z}$-covering. Then a metric completion $\tilde{M}_c$ **admits a structure of a complex manifold,** compatible with the complex structure on $\tilde{M} \subset \tilde{M}_c$.

For the proof see the end of this lecture.

DEFINITION: In assumptions of Theorem 2, the manifold $\tilde{M}_c$ is called **the cone** of an LCK manifold with potential.

Cone of an LCK manifold with potential.

Theorem 2: Let M be an LCK manifold with potential, and $\tilde{M}$ its Kähler $\mathbb{Z}$-covering. Then a metric completion $\tilde{M}_c$ **admits a structure of a complex manifold,** compatible with the complex structure on $\tilde{M} \subset \tilde{M}_c$.

Claim 1: The complement $\tilde{M}_c \backslash \tilde{M}$ is just one point, called **the origin**.

Proof: Indeed, let $z_i = \gamma^{n_i}(x_i)$ be a sequence of points in $\tilde{M}$, with each x_i in the fundamental domain $\varphi^{-1}([1, \lambda])$ of the $\Gamma = \mathbb{Z}$-action. Clearly, the distance between two fundamental domains $M_n := \gamma^n \varphi^{-1}([1, \lambda]) = \varphi^{-1}([\lambda^n, \lambda^{n+1}])$ and $M_{n+k+2} = \gamma^{n+k+2}\varphi^{-1}([1, \lambda])$ is written as

$$d(M_n, M_{n+k+2}) = \sum_{i=0}^{k} \lambda^{n+i} v, \qquad (*)$$

where v is a distance between M_0 and M_2. Then, z_i may converge only if $\lim_i n_i = -\infty$ or if all n_i, except finitely many, belong to a set $(p, p+1)$ for some p. The second case is irrelevant, because each M_i is compact, and in the first case, $\{z_i\}$ is always a Cauchy sequence, as follows from (*). All such $\{z_i\}$ are therefore equivalent, hence **converge to the same point in the metric completion.** ■

Normal families of functions

DEFINITION: Let M be a complex manifold, and $\mathcal{F}$ a family of holomorphic functions $f_i \in H^0(\mathcal{O}_M)$. We call $\mathcal{F}$ **a normal family** if for each compact $K \subset M$ there exists $C_K > 0$ such that for each $f \in \mathcal{F}$, $\sup_K |f| \leqslant C_K$.

LEMMA: Let M be a complex Hermitian manifold, $\mathcal{F} \subset H^0(\mathcal{O}_M)$ a normal family, and $K \subset M$ a compact. **Then there exists a number $A_K > 0$ such that $\sup_K |f'| \leqslant A_K$.**

Proof: Assume otherwise. Then there exists $x \in K$, $v \in T_x M$, and a sequence $f_i \in \mathcal{F}$ such that $\lim_i |D_v f_i| = \infty$. Pick a disk $\Delta \stackrel{j}{\hookrightarrow} M$ with compact closure in M, tangent to v in x, such that $j(0) = x$. Let $w = tv$ be the unit tangent vector. Then $\sup_\Delta |f_i| < C_\Delta$ by the normal family condition. **By Schwartz lemma, this implies $|D_w f_i| < C_\Delta$.** However, $t^{-1} \lim_i |D_w f_i| = \lim_i |D_v f_i| = \infty$ – contradiction. ■

C^0-topology on functions

DEFINITION: Let $C(M)$ be the space of functions on a topological space. **Topology of uniform convergence on compacts** (also known as **compact-open topology**; usually denoted as C^0) is a topology on $C(M)$ where a base of open sets is given by

$$U(X,C) := \{f \in C(M) \ \mid \ \sup_K |f| < C\},$$

for all compacts $K \subset M$ and $C > 0$. A sequence f_i of functions converges to f if it converges to f uniformly on all compacts.

REMARK: When M is locally compact, any sequence of continuous functions converging in C^0 **converges to a continuous function (prove it!)**

REMARK: In a similar way one defines **C^0-topology on the space of sections of a bundle.**

C^1-topology

DEFINITION: Let B be a vector bundle on a smooth manifold M, and $\nabla: B \longrightarrow B \otimes \Lambda^1 M$ a connection. Define **C^1-topology** on the space of sections of B (denoted, as usual, by the same letter B) as one where a subbase of open sets is given by C^0-open sets on B and $\nabla^{-1}(W)$, where W is an open set in C^0-topology in $B \otimes \Lambda^1 M$.

REMARK: A sequence f_i converges in C^1-topology if it converges uniformly on all compacts, and first derivatives f_i' also converge uniformy on all compacts. **This can be seen as a definition of C^1-topology.**

EXERCISE: Prove that C^1-topology is independent from the choice of a connection.

EXERCISE: Prove that the topological vector space $C^1 M$ of 1-differentialble functions on a manifold **is complete in C^1-topology.**

Arzelà-Ascoli theorem for normal families

THEOREM: Let M be a complex manifold and $\mathcal{F} \subset H^0(\mathcal{O}_M)$ a normal family of functions. Denote by $\overline{\mathcal{F}}$ its closure in C^0-topology. **Then $\overline{\mathcal{F}}$ is compact and contained in $H^0(\mathcal{O}_M)$.**

Proof. Step 1: Let $\{f_i\}$ be a sequence of functions in $\mathcal{F}$. By Tychonoff theorem, for each compact K, there is a subsequence of $\{f_i\}$ which converges pointwise on a dense countable subset $Z \subset K$. Taking diagonal subsequence, we find a subsequence $\{f_{p_i}\} \subset \{f_i\}$ which converges pointwise on a dense countable subset $Z \subset M$. Since $|f_i'|$ is uniformly bounded on compacts, the limit $f := \lim_i f_i$ is Lipschitz on all compact subsets of M. **Then it is continuous,** because a pointwise limit of Lipschitz functions is again Lipschitz.

Step 2: Since $|f_i'|$ is uniformly bounded on compacts, we can assume that f_i' also converges pointwise in Z, and $f := \lim_i f_i$ is differentiable. Since a limit of complex-linear operators is complex linear, Df is complex linear, and f is holomorphic. **This implies that $\overline{\mathcal{F}} \cap H^0(\mathcal{O}_M)$ is compact.** ■

Metric completion of a $\mathbb{Z}$-covering

THEOREM: (Rossi 1965, Andreotti-Siu 1970)
Let M be a complex manifold with boundary, $\dim_{\mathbb{C}} M > 2$, and φ a proper Kähler potential on M, taking values in $[a, \infty[$, and equal to a in the boundary of M. **Then there exists a Stein variety M_a** with isolated singularities, obtained by gluing a compact domain to M, and it is unique. Moreover, any holomorphic function on M can be extended to M_a.

Theorem 2: Let M be an LCK manifold with potential $\varphi: \tilde{M} \longrightarrow \mathbb{R}^{>0}$, where $\tilde{M}$ is its Kähler $\mathbb{Z}$-covering. Then a metric completion $\tilde{M}_c$ **admits a structure of a complex manifold,** compatible with the complex structure on $\tilde{M} \subset \tilde{M}_c$.

Proof. Step 1: Apply Rossi-Andreotti-Siu to $\varphi^{-1}([a, \infty[)$, we obtain a Stein variety $\tilde{M}_a$ containing $\varphi^{-1}([a, \infty[)$. Since $\tilde{M}_a$ contains $\varphi^{-1}([a_1, \infty[)$ for any $a_1 > a$, and the Rossi-Andreotti-Siu variety is unique, one has $\tilde{M}_a = \tilde{M}_{a_1}$. This implies that $\tilde{M}_a =: \tilde{M}_c$ **is independent from the choice of $a \in \mathbb{R}^{>0}$.**

It remains to identify $\tilde{M}_c$ with a metric completion of $\tilde{M}$. By Claim 1, **this is equivalent to the complement $\tilde{M}_c \backslash \tilde{M}$ being a one-point set.**

Metric completion of a $\mathbb{Z}$-covering (part 2)

Theorem 2: Let M be an LCK manifold with potential $\varphi: \tilde{M} \longrightarrow \mathbb{R}^{>0}$, where $\tilde{M}$ is its Kähler $\mathbb{Z}$-covering. Then a metric completion $\tilde{M}_c$ **admits a structure of a complex manifold,** compatible with the complex structure on $\tilde{M} \subset \tilde{M}_c$.

It remains to identify $\tilde{M}_c$ with a metric completion of $\tilde{M}$. By Claim 1, **this is equivalent to the complement $\tilde{M}_c \backslash \tilde{M}$ being a one-point set.**

Step 2: The monodromy group $\Gamma = \mathbb{Z}$ acts on $\tilde{M}_c$ by holomorphic automorphisms. Indeed, any holomorphic function (hence, any holomorphic map) can be extended from $\tilde{M}$ to $\tilde{M}_c$ uniquely.

Step 3: Denote by γ the generator of Γ which decreases the metric by $\lambda < 1$, and let $\tilde{M}_c^a$ be a Stein variety associated with $\varphi^{-1}(]0, a]) \subset \tilde{M}$ as above. Since $\gamma(\tilde{M}_c^a) = \tilde{M}_c^{\lambda a}$, for any holomorphic function f on $\tilde{M}_c$, one has

$$\sup_{z \in \tilde{M}_c^a} |f(\gamma^n(z))| = \sup_{z \in \tilde{M}_c^{\lambda^n a}} |f(z)| \leqslant \sup_{z \in \tilde{M}_c^a} |f(z)|.$$

Therefore, $\{f(\gamma^n(z))\}$ is a normal family.

Metric completion of a $\mathbb{Z}$-covering (part 3)

Step 3: Denote by γ the generator of Γ which decreases the metric by $\lambda < 1$, and let $\tilde{M}_c^a$ be a Stein variety associated with $\varphi^{-1}(]0, a]) \subset \tilde{M}$ as above. Since $\gamma(\tilde{M}_c^a) = \tilde{M}_c^{\lambda a}$, for any holomorphic function f on $\tilde{M}_c$, one has

$$\sup_{z \in \tilde{M}_c^a} |f(\gamma^n(z))| = \sup_{z \in \tilde{M}_c^{\lambda^n a}} |f(z)| \leqslant \sup_{z \in \tilde{M}_c^a} |f(z)|.$$

Therefore, $\{f(\gamma^n(z))\}$ is a normal family.

Step 4: Let f_{lim} be any limit point of the sequence $\{f(\gamma^n(z))\}$. Since the sequence $t_i := \sup_{z \in \tilde{M}_c^{\lambda^i a}} |f(z)|$ is non-increasing, it converges, and $\sup_{z \in \tilde{M}_c^a} f_{\text{lim}} = \lim t_i$. Similarly, $\sup_{z \in \tilde{M}_c^{\lambda a}} f_{\text{lim}} = \lim t_i$. By strong maximum principle, **a non-constant holomorphic function on a complex manifold with boundary cannot have local maxima (even non-strict) outside of the boundary.** Since $\tilde{M}_c^{\lambda a}$ does not intersect the boundary of $\tilde{M}_c^a$, the function f_{lim} must be constant.

Step 5: Consider now the complement $V := \tilde{M}_c \backslash \tilde{M}$, and suppose it has two distinct points x and y. Let f be a holomorphic function which satisfy $f(x) \neq f(y)$. Replacing f by an exponent of μf if necessarily, we may assume that $|f(x)| < |f(y)$. Since γ fixes Z, which is compact, **for any limit f_{lim} of the sequence $\{f(\gamma^n(z))\}$, supremum f_{lim} on Z is not equal to infimum of f_{lim} on Z.** This is impossible, hence $f = const$ on V, and V is one point. ■