Locally conformally Kähler manifolds

lecture 9: holomorphic contractions and Riesz-Schauder theorem

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LCK manifolds (reminder)

DEFINITION: Let (M, I, ω) be a Hermitian manifold, $\dim_{\mathbb{C}} M > 1$. Then M is called **locally conformally Kähler** (LCK) if $d\omega = \omega \wedge \theta$, where θ is a closed 1-form, called **the Lee form**.

DEFINITION: A manifold is locally conformally Kähler iff it admits a Kähler form taking values in a positive, flat vector bundle L, called **the weight bundle**.

DEFINITION: Deck transform, or **monodromy maps** of a covering $\tilde{M} \longrightarrow M$ are elements of the group $\operatorname{Aut}_M(\tilde{M})$. **When $\tilde{M}$ is a universal cover, one has $\operatorname{Aut}_M(\tilde{M}) = \pi_1(M)$.**

DEFINITION: An LCK manifold is a complex manifold such that its universal cover $\tilde{M}$ is equipped with a Kähler form $\tilde{\omega}$, and the deck transform acts on $\tilde{M}$ by Kähler homotheties.

THEOREM: These three definitions are equivalent.

Conical Kähler manifolds (reminder)

DEFINITION: Let (X, g) be a Riemannian manifold, and $C(X) := X \times \mathbb{R}^{>0}$, with the metric $t^2 g + dt^2$, where t is a coordinate on $\mathbb{R}^{>0}$. Then $C(X)$ is called **Riemannian cone** of X. **Multiplicative group $\mathbb{R}^{>0}$ acts on $C(X)$ by homotheties, $(m, t) \longrightarrow (m, \lambda t)$.**

DEFINITION: Let (X, g) be a Riemannian manifold, $C(X) := X \times \mathbb{R}^{>0}$ its Riemannian cone, and h_λ the homothety action. Assume that (X, g) is equipped with a complex structure, in such a way that g is Kähler, and h_λ acts holomorphically. Then $C(X)$ is called **a conical Kähler manifold**. In this situation, X is called **Sasakian manifold**.

REMARK: A **contact manifold** is defined as a manifold X with symplectic structure on $C(X)$, and h_λ acting by homotheties. In particular, **Sasakian manifolds are contact**. **Sasakian geometry is an odd-dimensional counterpart to Kähler geometry**

EXAMPLE: Let L be a positive holomorphic line bundle on a projective manifold. **Then the total space of its unit S^1-fibration is Sasakian.**

Kähler potentials and plurisubharmonic functions (reminder)

DEFINITION: A real-valued smooth function on a complex manifold is called **plurisubharmonic (psh)** if the (1,1)-form $dd^c f$ is positive, and **strictly plurisubharmonic** if $dd^c f$ is an Hermitian form.

REMARK: Since $dd^c f$ is always closed, **it is also Kähler when it is strictly positive.**

DEFINITION: Let (M, I, ω) be a Kähler manifold. **Kähler potential** is a function f such that $dd^c f = \omega$.

THEOREM: Let S be a Sasakian manifold, $C(S) = S \times \mathbb{R}^{>0}$ its cone, t the coordinate along the second variable, and $r = t\frac{d}{dt}$. **Then t^2 is a Kähler potential on $C(S)$.**

LCK manifolds with potential (reminder)

DEFINITION: Let M be an LCK manifold, and $(\tilde M, \tilde\omega)$ its Kähler covering. It is called **LCK manifold with potential** if $\tilde M$ admits an automorphic Kähler potential $\varphi: \tilde M \longrightarrow \mathbb{R}^{>0}$, $dd^c\varphi = \tilde\omega$, which is **proper** (preimage of a compact is again compact).

THEOREM: The property of being LCK with potential is stable under small deformations.

REMARK: For any complex submanifold $Z \subset M$ of an LCK manifold with potential, Z is also an LCK manifold with potential.

THEOREM: Let M be an LCK manifold, $\Gamma \subset \mathbb{R}^{>0}$ the monodromy group, and $(\tilde M, \tilde\omega)$ its Kähler covering, with $\tilde M/\Gamma = M$. Assume that $\tilde\omega$ admits a Γ-automorphic Kähler potential φ. **The map φ is proper if and only if $\Gamma = \mathbb{Z}$.**

THEOREM: Let M be an LCK manifold with potential, and $\tilde M$ its Kähler $\mathbb{Z}$-covering. Then a metric completion $\tilde M_c$ **admits a structure of a complex manifold,** compatible with the complex structure on $\tilde M \subset \tilde M_c$. Moreover, the monodromy action on $\tilde M$ is extended to a holomorphic automorphism of $\tilde M_c$.

Normal families of functions (reminder)

DEFINITION: Let $C(M)$ be the space of functions on a topological space. **Topology of uniform convergence on compacts** (also known as **compact-open topology**; usually denoted as C^0) is a topology on $C(M)$ where a base of open sets is given by

$$U(X, C) := \{f \in C(M) \mid \sup_K |f| < C\},$$

for all compacts $K \subset M$ and $C > 0$. A sequence f_i of functions converges to f if it converges to f uniformly on all compacts.

DEFINITION: Let M be a complex manifold, and $\mathcal{F}$ a family of holomorphic functions $f_i \in H^0(\mathcal{O}_M)$. We call $\mathcal{F}$ **a normal family** if for each compact $K \subset M$ there exists $C_K > 0$ such that for each $f \in \mathcal{F}$, $\sup_K |f| \leqslant C_K$.

THEOREM: Let M be a complex manifold and $\mathcal{F} \subset H^0(\mathcal{O}_M)$ a normal family of functions. Denote by $\overline{\mathcal{F}}$ its closure in C^0-topology. **Then $\overline{\mathcal{F}}$ is compact and contained in $H^0(\mathcal{O}_M)$.**

Banach space of holomorphic functions

DEFINITION: A **Banach space** is a complete normed vector space.

THEOREM: Let M be a complex manifold, and $H^0_b(\mathcal{O}_M)$ the space of all bounded holomorphic functions, equipped with **the sup-norm** $|f|_{\sup} := \sup_M |f|$. **Then it is a Banach space.**

Proof. Step 1: Let $\{f_i\} \in H^0_b(\mathcal{O}_M)$ be a Cauchy sequence in sup-norm. **Then $\{f_i\}$ converges to a continuous function f** in sup-topology.

Step 2: Since $\{f_i\}$ is a normal family, it has a subsequence which converges in C^0-topology to $\tilde f \in H^0(\mathcal{O}_M)$. However, the **$C^0$-topology is weaker than the sup-topology, hence $\tilde f = f$.** Therefore, f is holomorphic. ■

Compact operators

DEFINITION: A **bounded subset** of a topological vector space V is a set $B \subset V$ such that for any open neighbourhood $U \ni 0$, there exists $\lambda > 0$ such that $\lambda B \subset U$

REMARK: Bounded subsets of normed spaces are subsets which are contained in a ball of a sufficiently big radius.

DEFINITION: A subset of a topological space is called **precompact** if its closure is compact.

DEFINITION: Let V, W be topological vector spaces, and $\varphi : V \longrightarrow W$ a continuous linear operator. It is called **compact** if an image of any bounded set is precompact.

EXERCISE: Let $V = H^0(\mathcal{O}_M)$ be a space of holomorphic functions on a complex manifold M with C^0-topology. **Prove that any bounded subset of V is precompact.** In this case, the identity map is a compact operator.

REMARK: By Riesz theorem, **a closed ball in a normed vector space V is never compact,** unless V is finite-dimensional. This means that $(H^0(\mathcal{O}_M), C^0)$ does not admit a norm.

Holomorphic contractions

DEFINITION: Contraction of a manifold M to a point $x \subset M$ is a morphism $\varphi: M \longrightarrow M$ such that for any compact subset $K \subset M$ and any open $U \ni x$ there exists $N > 0$ such that for all $n > N$, the map φ^n maps K to U.

THEOREM: Let X be a complex variety, and $\gamma: X \longrightarrow X$ a holomorphic contraction such that $\gamma(X)$ is precompact. Consider the Banach space $V = H^0_b(\mathcal{O}_X)$ with sup-metric. **Then $\gamma^*: V \longrightarrow V$ is compact,** and **its operator norm $\|\gamma^*\| := \sup_{|v|\leqslant 1}|\gamma^*(v)|$ is strictly less than 1.**

Proof. Step 1: Let $B_C := \{v \in V \mid |v|_{\sup} \leqslant C\}$. Then

$$|\gamma^* f|_{\sup} = \sup_{x\in\overline{\gamma(X)}}|f(x)|.$$

Therefore, **for any sequence $\{f_i\}$ converging in C^0-topology, the sequence $\{\gamma^* f_i\}$ converges in sup-topology.** However, B_C is precompact in C^0-topology, because it is a normal family. Then $\gamma^* B_C$ is precompact in sup-topology.

Holomorphic contractions (part 2)

EXERCISE: Prove **the maximum principle**: **a non-constant holomorphic function cannot have any non-strict maxima.**

Step 2: Since $\sup_X |\gamma^* f| = \sup_{\gamma(X)} |f| \leqslant \sup_X |f|$, one has $\|\gamma^*\| \leqslant 1$. If this inequality is not strict, for some sequence $f_i \in B_1$ one has $\lim_i \sup_{x\in\gamma(X)} |f_i(x)| = 1$. Since B_1 is a normal family, f_i has a subsequence converging in C^0-topology to f. Then $\gamma(f_i)$ converges to $\gamma(f)$ in sup-topology, giving $\lim_i \sup_{x\in\gamma(X)} |f_i(x)| = \sup_{x\in\gamma(X)} |f(x)| = 1$. **Since a holomorphic functions has no strict maxima, this means that $|f(x)| > 1$ somewhere on X.** Then f cannot be a C^0-limit of $f_i \in B_1$. ■

Hopf manifolds and finite vectors

DEFINITION: Let $A \in \operatorname{End}(\mathbb{C}^n)$ be an invertible linear endomorphism with all eigenvalues $|\alpha_i| < 1$. The quotient $H := (\mathbb{C}^n \backslash 0)/\langle A\rangle$ is called **a linear Hopf manifold**.

Theorem 1: Let M be an LCK manifold with potential, $\dim_{\mathbb{C}} M > 2$. **Then M admits a holomorphic embedding to a linear Hopf manifold.**

DEFINITION: Let γ be an endomorphism of a vector space V. A vector $v \in V$ is called **γ-finite** if the space $\langle v, \gamma(v), \gamma^2(v), ...\rangle$ is finite-dimensional.

Theorem 2: Let M be an LCK manifold with potential, $\dim_{\mathbb{C}} M > 2$, and $\tilde M$ its Kähler $\mathbb{Z}$-covering. Consider a metric completion $\tilde M_c$ with its complex structure and a contraction $\gamma: \tilde M_c \longrightarrow \tilde M_c$ generating the $\mathbb{Z}$-action. Let $H^0(\mathcal{O}_{\tilde M_c})_f$ be the space of functions which are γ^*-finite. **Then $H^0(\mathcal{O}_{\tilde M_c})_f$ is dense in sup-topology on each compact subset of $\tilde M_c$.**

We deduce Theorem 1 from Theorem 2, and then prove Theorem 2 using Riesz-Schauder Theorem.

Embedding into Hopf manifolds

Theorem 2 $\Rightarrow$ Theorem 1. Step 1:
Let $W \subset H^0(\mathcal{O}_{\tilde{M}_c})_f$ be an m-dimensional γ^*-invariant subspace W with basis $w_1, ..., w_m$. Then the following diagram is commutative:

$$\begin{array}{ccc} \tilde{M} & \stackrel{\Psi}{\longrightarrow} & \mathbb{C}^m \\ \gamma \Big\downarrow & & \Big\downarrow \gamma^*; \\ M & \stackrel{\Psi}{\longrightarrow} & \mathbb{C}^m \end{array}$$

where $\Psi(x) = (w_1(x), w_2(x), ..., w_m(x))$.

Suppose that the map Ψ associated with a given $W \subset H^0(\mathcal{O}_{\tilde{M}_c})_f$ is injective. Then **the quotient map gives an embedding $\Psi: \tilde{M}/\mathbb{Z} \longrightarrow (\mathbb{C}^m \backslash 0)/\gamma^*$;** all eigenvalues of γ^* are < 1 because its operator norm is < 1.

Step 2: To find an appropriate $W \subset H^0(\mathcal{O}_{\tilde{M}_c})_f$, choose a holomorphic embedding $\Psi_1 : \tilde{M}_c \hookrightarrow \mathbb{C}^n$, which exists because $\tilde{M}_c$ is Stein. Let $\tilde{w}_1, ..., \tilde{w}_n$ be the coordinate functions of Ψ_1. Theorem 2 allows one to approximate $\tilde{w}_i$ by $w_i \in H^0(\mathcal{O}_{\tilde{M}_c})_f$ in C^0-topology. **Choosing w_i sufficiently close to $\tilde{w}_i$ in a compact fundamental domain of $\mathbb{Z}$-action, we obtain that $x \longrightarrow (w_1(x), w_2(x), ..., w_n(x))$ is injective in a compact fundamental domain of $\mathbb{Z}$.** To finish the argument, take $W \subset H^0(\mathcal{O}_{\tilde{M}_c})_f$ generated by the γ^* from $w_1, ..., w_n$, and apply Step 1. ■

Riesz-Schauder Theorem

To find enough γ^*-finite vectors, we use the Riesz-Schauder Theorem. It is a Banach analogue of spectral theorem which easily follows from Fredholm theory.

THEOREM: (Riesz-Schauder)
Let $F : V \longrightarrow V$ be a compact operator on a Banach space. Then for each non-zero $\mu \in \mathbb{C}$, there exists a sufficiently big number $N \in \mathbb{Z}$ such that for each $n > N$ **one has** $V = \ker(F - \mu \operatorname{Id})^n \oplus \overline{\operatorname{im}(F - \mu \operatorname{Id})^n}$**, where** $\overline{\operatorname{im}(F - \mu \operatorname{Id})^n}$ **is closure of the image.** Moreover, $\ker(F - \mu \operatorname{Id})^n$ is finite-dimensional and independent on n.

Riesz-Schauder Theorem and adic filtration

In our case the Riesz-Schauder theorem is especially effective.

Proposition 1: Fix a precompact subset $\tilde{M}_c^a := \varphi^{-1}([0, a[)$, where $\varphi : \tilde{M}_c$ is the Kähler potential. Let A be the ring of bounded holomorphic functions on $\tilde{M}_c^a$, and $\mathfrak{m}$ the maximal ideal of the origin point. Clearly, γ^* preserves $\mathfrak{m}$ and all its powers. Let $P_k(t)$ be the minimal polynomial of $\gamma^*\Big|_{A/\mathfrak{m}^k}$. **Then** $\operatorname{im}(P_k(\gamma^*) \subset \mathfrak{m}^k(A)$**, and** $\ker P_k(\gamma^*)$ **generates** $A/\mathfrak{m}^k$**.**

Proof: Since $P_k(t)$ is a minimal polynomial of γ^* on $A/\mathfrak{m}^k$, the endomorphism $P_k(\gamma^*)$ acts trivially on $A/\mathfrak{m}^k$, hence it maps A to $\mathfrak{m}^k$.

From Riesz-Schauder theorem applied to the Banach space A and $F = P_k(\gamma^*)$ it follows that $A = \operatorname{im}(P_k(\gamma^*) + \overline{\operatorname{im}(F - \mu \operatorname{Id})^n}$ hence $\ker P_k(\gamma^*)$ generates $A/\mathfrak{m}^k$.
■

$\mathfrak{m}$-adic topology and C^0-topology

DEFINITION: Let A be a ring and $\mathfrak{m}$ an ideal. The base of open sets in **$\mathfrak{m}$-adic topology on A** are $\mathfrak{m}^k$ and their translates.

Proposition 1 implies the following result

PROPOSITION: Let $H^0(\mathcal{O}_{\tilde{M}_c})_f \subset H^0(\mathcal{O}_{\tilde{M}_c})$ be the set of γ^*-finite functions and $\mathfrak{m}$ the maximal ideal of the origin in $\tilde{M}_c$. **Then $H^0(\mathcal{O}_{\tilde{M}_c})_f$ is dense in $\mathfrak{m}$-adic topology.**

Proof: A subspace $V \subset A$ is dense in $\mathfrak{m}$-adic topology in $A \Leftrightarrow$ the quotient $V/v \cap \mathfrak{m}^k$ surjects to $A/\mathfrak{m}^k$. This is proven in Proposition 1 for the ring of bounded holomorphic functions on $\tilde{M}_c^a$. However, any such function can be extended to γ^*-finite function on $\tilde{M}_c$ using γ^*-action. ■

Now Theorem 2 is implied by the following claim.

CLAIM: Let X be a connected complex variety, A the ring of bounded holomorphic functions, $x \in X$ a point, $\mathfrak{m} \subset A$ its maximal ideal, and $R: A \longrightarrow \hat{A}$ the natural map from A to its $\mathfrak{m}$-adic completion. **Then R is continuous in C^0-topology and induces homeomorphism of any bounded set to its image.**

$\mathfrak{m}$-adic topology and C^0-topology, part 2

CLAIM: Let X be a connected complex variety, A the ring of bounded holomorphic functions, $x \in X$ a point, $\mathfrak{m} \subset A$ its maximal ideal, and $R: A \longrightarrow \hat{A}$ the natural map from A to its $\mathfrak{m}$-adic completion. **Then R is continuous in C^0-topology and induces homeomorphism of any bounded set to its image.**

Proof: Continuity is clear because C^0-topology is equivalent to C^1-topology, C^2-topology and so on (Lecture 8). Therefore, taking successive derivatives in a point is continuous in C^0-topology. However, R takes a function and replaces it by its Taylor series.

To see that R is a homeomorphism, notice that any bounded, closed subset of A is compact, hence its image under a continuous map is also closed. ■